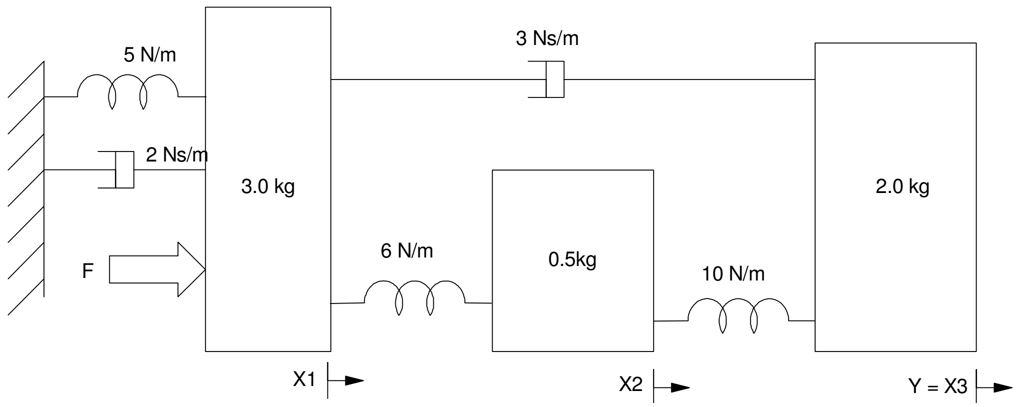
Homework #5: ECE 461/661

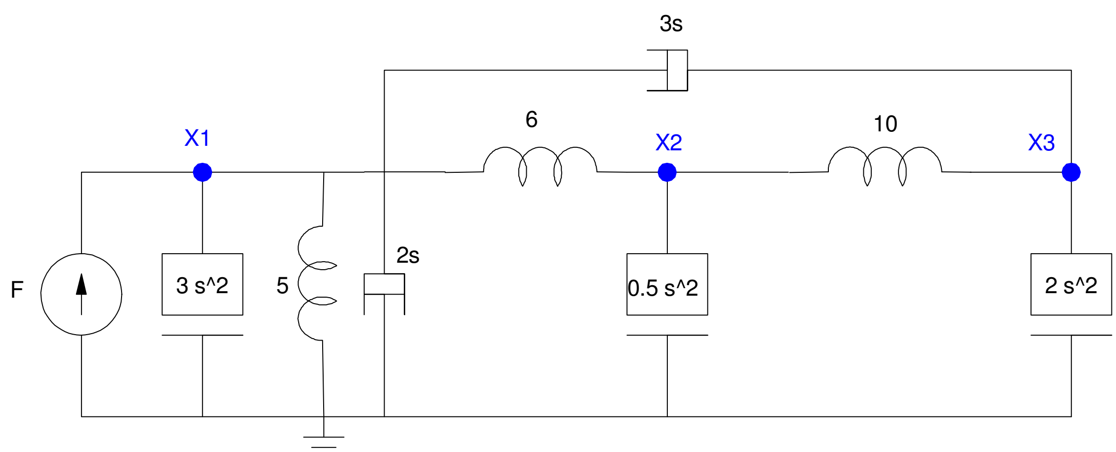
Mass-Spring Systems, Rotational Systems. Due Monday, September 27th

Mass Spring systems

1) For the following mass-spring system:



Draw the circuit equivalent for the following mass-spring systems.



Express the dynamics in state-space form

$$(3s^2 + 2s + 5 + 6 + 3s)X_1 - (6)X_2 - (3s)X_3 = F$$

$$(0.5s^2 + 6 + 10)X_2 - (6)X_1 - (10)X_3 = 0$$

$$(2s^2 + 3s + 10)X_3 - (3s)X_1 - (10)X_2 = 0$$

Solve for the highest derivative

$$s^2X_1 = \left(-\tfrac{5}{3}s - \tfrac{11}{3}\right)X_1 + 2X_2 + sX_3 + \tfrac{1}{3}F$$

$$s^2X_2 = -32X_2 + 12X_1 + 20X_3$$

$$s^2X_3 = \left(-\tfrac{3}{2}s - 5\right)X_3 + \tfrac{3}{2}sX_1 + 5X_2$$

Place in matrix form

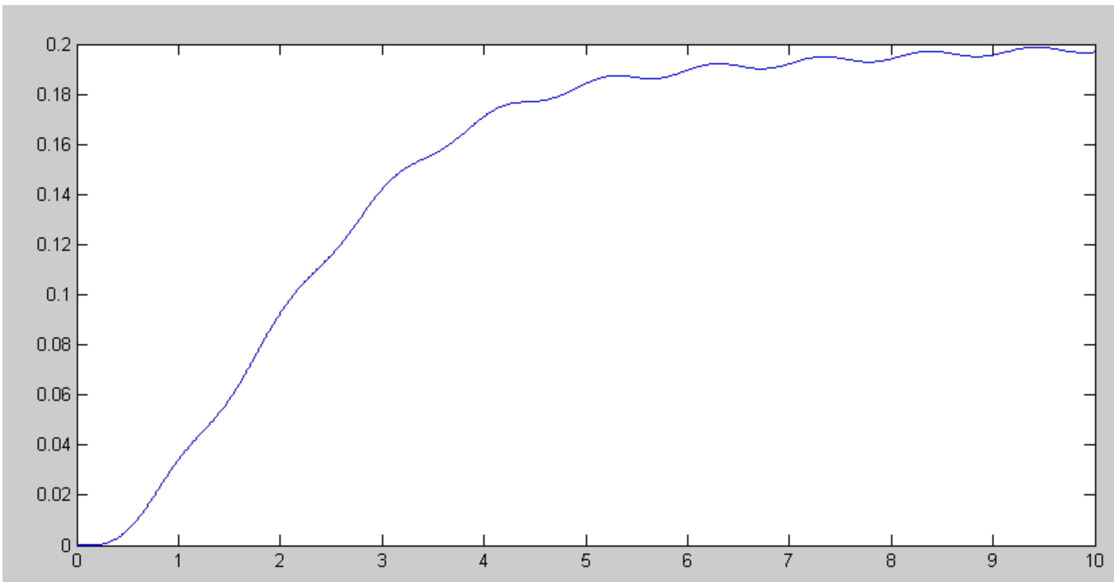
$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -\tfrac{11}{3} & 2 & 0 & \vdots & -\tfrac{5}{3} & 0 & 1 \\ 12 & -32 & 20 & \vdots & 0 & 0 & 0 \\ 0 & 5 & -5 & \vdots & \tfrac{3}{2} & 0 & -\tfrac{3}{2} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ \tfrac{1}{3} \\ 0 \\ 0 \end{bmatrix}F$$

Find the transfer function from F to X2

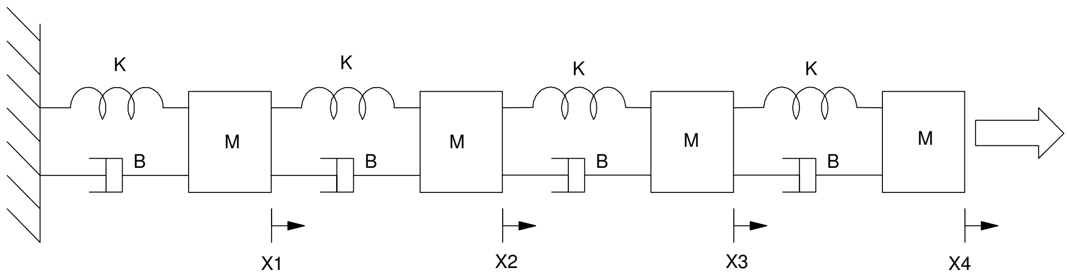
```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-11/3,2,0 ; 12,-32,20 ; 0,5,-5];
>> a22 = [-5.3,0,1 ; 0,0,0 ; 3/2,0,-3/2];
>> A = [a11,a12 ; a21,a22]

A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -3.6667    2.0000         0   -5.3000         0    1.0000
   12.0000  -32.0000   20.0000         0         0         0
         0    5.0000   -5.0000    1.5000         0   -1.5000

>> B = [0;0;0;1/3;0;0];
>> C = [0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

Zero/pole/gain:
                        4 (s^2 + 4s + 5)
-----------------------------------------------------------------
(s+4.91) (s+0.4769) (s^2 + 1.283s + 1.193) (s^2 + 0.1298s + 35.8)
```

Plot the step response from F to X2

```
>> t = [0:0.01:10]';
>> y = step(G,t);
>> plot(t,y)
>> eig(A)

  -0.0649 + 5.9832i
  -0.0649 - 5.9832i
  -4.9099
  -0.4769
  -0.6417 + 0.8838i
  -0.6417 - 0.8838i

>>
```

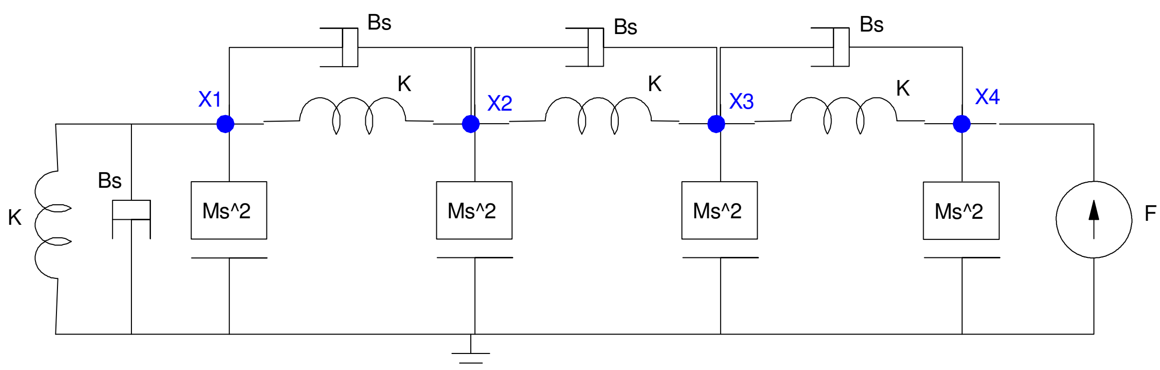


2) For the following mass-spring system...



Problem 6: M = 2.0kg, B = 0.3 Ns/m, K = 10 N/m

Draw the circuit equivalent



Express the dynamics in state-space form plug in (K/M = 5), (B/M = 0.15)

$$(Ms^2 + 2Bs + 2K)X_1 - (Bs + K)X_2 = 0$$

$$(Ms^2 + 2Bs + 2K)X_2 - (Bs + K)X_1 - (Bs + K)X_3 = 0$$

$$(Ms^2 + 2Bs + 2K)X_3 - (Bs + K)X_2 - (Bs + K)X_4 = 0$$

$$(Ms^2 + Bs + K)X_4 - (Bs + K)X_3 = F$$

Place in matix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \\ \cdots \\ s^2X_1 \\ s^2X_2 \\ s^2X_3 \\ s^2X_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\ \frac{-2K}{M} & \frac{K}{M} & 0 & 0 & \vdots & \frac{-2B}{M} & \frac{B}{M} & 0 & 0 \\ \frac{K}{M} & \frac{-2K}{M} & \frac{K}{M} & 0 & \vdots & \frac{B}{M} & \frac{-2B}{M} & \frac{B}{M} & 0 \\ 0 & \frac{K}{M} & \frac{-2K}{M} & \frac{K}{M} & \vdots & 0 & \frac{B}{M} & \frac{-2B}{M} & \frac{B}{M} \\ 0 & 0 & \frac{K}{M} & \frac{-K}{M} & \vdots & 0 & 0 & \frac{B}{M} & \frac{-B}{M} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix} F$$

plug into Matlab noting that

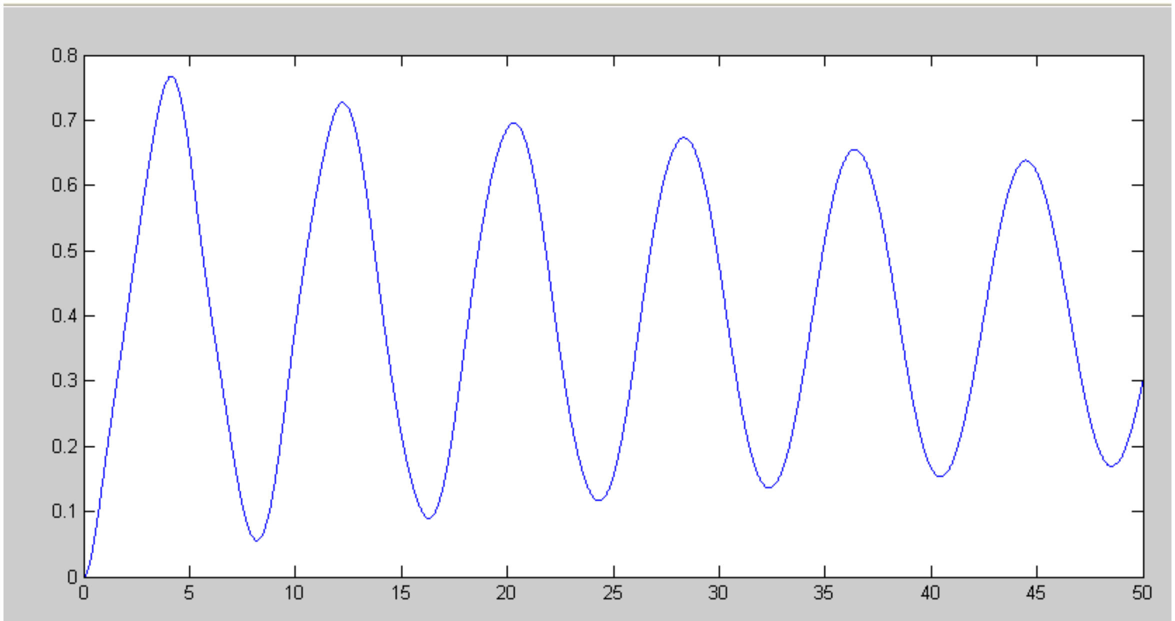
- K/M = 5
- B/M = 0.15
-

Find the transfer function from F to X4

```
>> a11 = zeros(4,4);
>> a12 = eye(4,4);
>> a21 = [-10,5,0,0 ; 5,-10,5,0 ; 0,5,-10,5 ; 0,0,5,-5];
>> a22 = [-0.3,0.15,0,0 ; 0.15,-0.3,0.15,0 ; 0,0.15,-0.3,0.15 ; 0,0,0.15,-0.15];
>> A = [a11,a12 ; a21,a22]

A =

         0         0         0         0    1.0000         0         0         0
         0         0         0         0         0    1.0000         0         0
         0         0         0         0         0         0    1.0000         0
         0         0         0         0         0         0         0    1.0000
  -10.0000    5.0000         0         0   -0.3000    0.1500         0         0
    5.0000  -10.0000    5.0000         0    0.1500   -0.3000    0.1500         0
         0    5.0000  -10.0000    5.0000         0    0.1500   -0.3000    0.1500
         0         0    5.0000   -5.0000         0         0    0.1500   -0.1500

>> B = [0;0;0;0;0;0;0;1/2]

B =

         0
         0
         0
         0
         0
         0
         0
    0.5000

>> C = [0,0,0,1,0,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
??? Error using ==> ss.ss>ss.ss at 345
The values of the "a" and "c" properties must be matrices with the same
number of columns.

>> C = [0,0,0,1,0,0,0,0];;
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


        0.5 (s^2 + 0.08787s + 2.929) (s^2 + 0.3s + 10) (s^2 + 0.5121s + 17.07)
------------------------------------------------------------------------------------
(s^2 + 0.01809s + 0.6031) (s^2 + 0.15s + 5) (s^2 + 0.3521s + 11.74) (s^2 + 0.5298s + 17.66)

>> eig(A)


  -0.2649 + 4.1941i
  -0.2649 - 4.1941i
  -0.1760 + 3.4213i
  -0.1760 - 3.4213i
  -0.0750 + 2.2348i
  -0.0750 - 2.2348i
  -0.0090 + 0.7765i
  -0.0090 - 0.7765i


>>
```

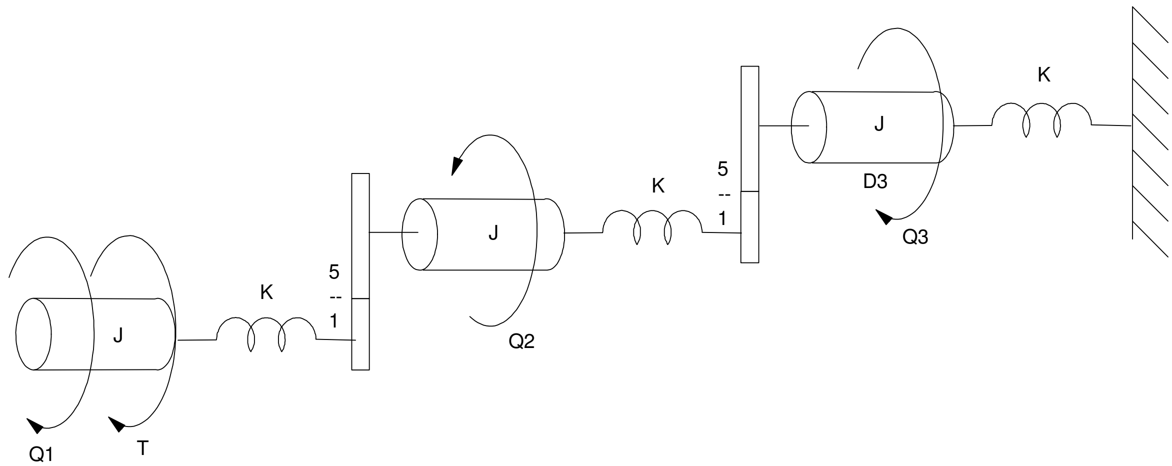
Plot the step response from F to X4

```
>> t = [0:0.01:50]';
>> y = step(G,t);
>> plot(t,y)
```



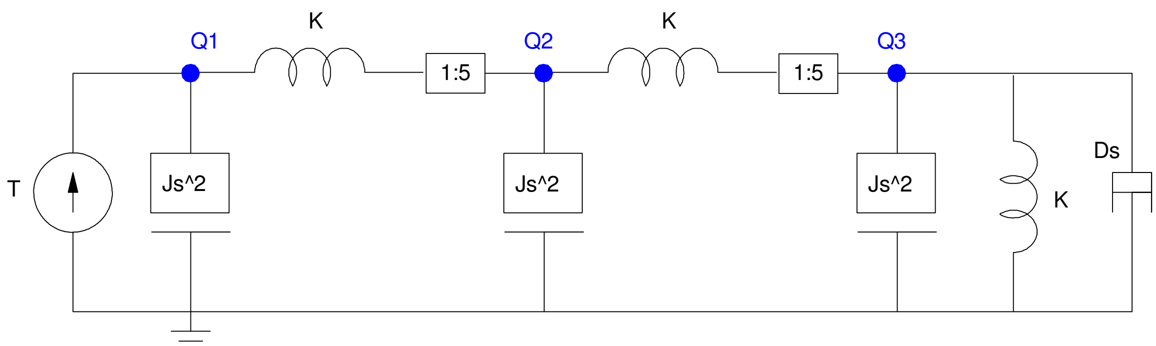
Rotational Systems

3) For the following rotational system...

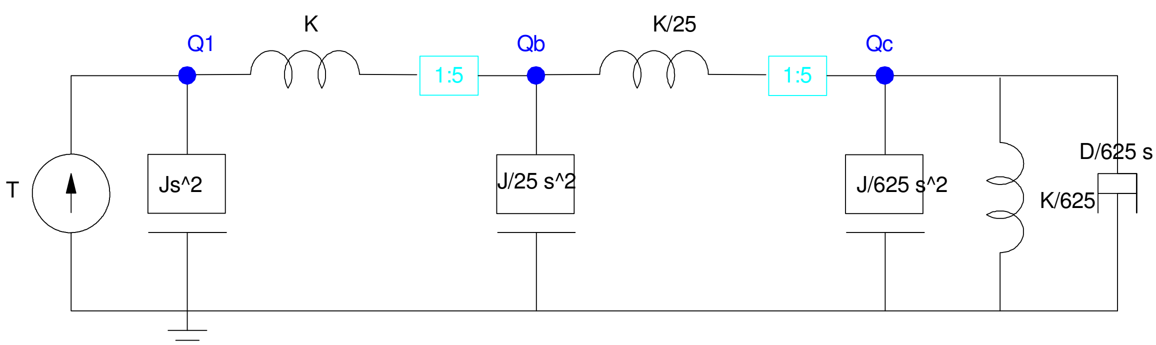


Problem 3: J = 2.5 Kg m / s^2. K = 20 Nm/rad, D3 = 1.5 Nms/ rad

Draw the circuit equivalent



Remove the gears



Write the node equations

$$(Js^2+K)\theta_1-(K)\theta_2=T$$

$$\left(\frac{J}{25}s^2+K+\frac{K}{25}\right)\theta_b-(K)\theta_1-\left(\frac{K}{25}\right)\theta_c=0$$

$$\left(\frac{J}{625}s^2+\frac{D}{625}s+\frac{K}{625}+\frac{K}{25}\right)\theta_c-\left(\frac{K}{25}\right)\theta_b=0$$

Solving for the highest derivative

Problem 3: J = 2.5 Kg m / s^2. K = 20 Nm/rad, D3 = 1.5 Nms/ rad

$$s^2\theta_1=\left(-\frac{K}{J}\right)\theta_1+\left(\frac{K}{J}\right)\theta_b+\left(\frac{1}{J}\right)T$$

$$s^2\theta_b=\left(\frac{25K}{J}\right)\theta_1-\left(\frac{26K}{J}\right)\theta_b+\left(\frac{K}{J}\right)\theta_c$$

$$s^2\theta_c=\left(\frac{25K}{J}\right)\theta_b-\left(\frac{D}{J}s+\frac{26K}{J}\right)\theta_c$$

Plugging in numbers and placing in matrix form (K/J = 8, D/J = 0.6)

$$\begin{bmatrix} s\theta_1 \\ s\theta_b \\ s\theta_c \\ \cdots \\ s^2\theta_1 \\ s^2\theta_b \\ s^2\theta_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -8 & 8 & 0 & \vdots & 0 & 0 & 0 \\ 200 & -208 & 8 & \vdots & 0 & 0 & 0 \\ 0 & 200 & -208 & \vdots & 0 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_b \\ \theta_c \\ \cdots \\ s\theta_1 \\ s\theta_b \\ s\theta_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0.4 \\ 0 \\ 0 \end{bmatrix} T$$

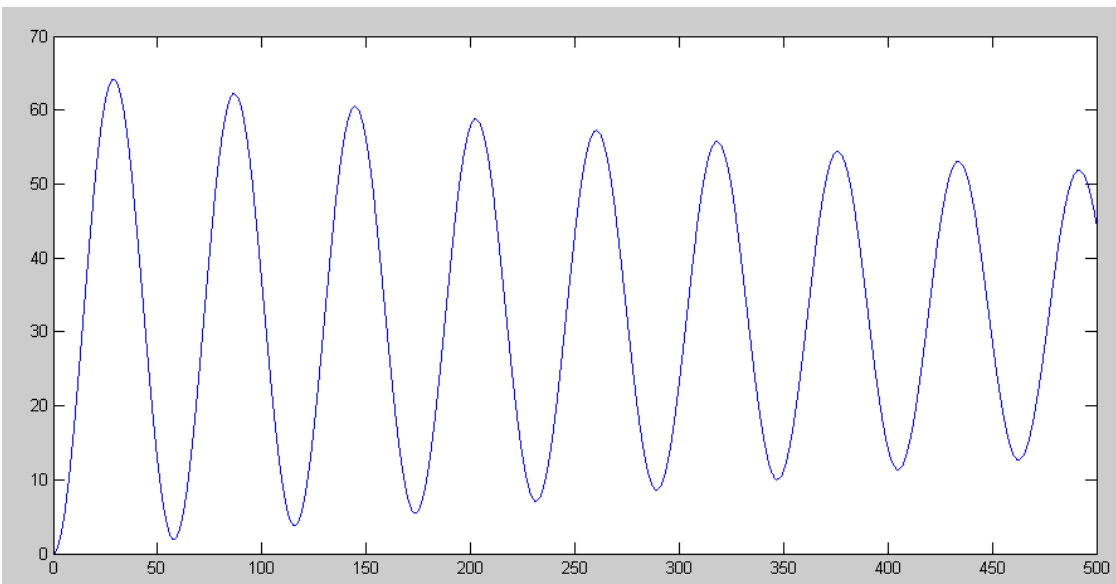
Find the transfer function from T to Q1

```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-8,8,0 ; 200,-208,8 ; 0,200,-208];
>> a22 = [0,0,0 ; 0,0,0 ; 0,0,-1.5];
>> A = [a11,a12 ; a21,a22]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
   -8.0000     8.0000          0          0          0          0
  200.0000  -208.0000     8.0000          0          0          0
         0   200.0000  -208.0000          0          0    -1.5000

>> B = [0;0;0;0.4;0;0];
>> C = [1,0,0,0,0,0];
>> D = 0;
```

```
>> G = ss(A,B,C,D);
>> zpk(G)

            0.4 (s^2 + 0.7555s + 169.2) (s^2 + 0.7445s + 246.2)
-------------------------------------------------------------------------
(s^2 + 0.002125s + 0.0118) (s^2 + 0.8275s + 173.8) (s^2 + 0.6703s + 249.6)

>> eig(A)

  -0.3352 +15.7959i
  -0.3352 -15.7959i
  -0.4138 +13.1771i
  -0.4138 -13.1771i
  -0.0011 + 0.1086i
  -0.0011 - 0.1086i

>>
```

Plot the step response from T to Q1

```
>> t = [0:0.1:500]';
>> y = step(G,t);
>> plot(t,y)
```



Motors

4) Find the transfer function for the following DC servo motor

http://www.baldor.com/catalog/CDP3306

Allen Bradley CDP3306: 1/4 HP Servo Motor

- $716ea
- Kt = 1.117 ft-lb @ 1.68A
- Ra = 8.54 Ohms
- La = 60.59mH
- J = 4.680 lb-ft2
- 1.43A @ 1690rpm @ 0.945 ft-lb load
- 0.1A @ 180V @ 1830rpm @ 0 ft-lb load
- Weight 23.0kg

$$K_t = \left(\frac{1.117\text{ ft lb}}{1.68A}\right)\left(\frac{1m}{3.281ft}\right)\left(\frac{4.445N}{lb}\right) = 0.9008\frac{Nm}{A}$$

$$J = 4.608(lb \cdot ft^2)\left(\frac{0.4536kg}{lb}\right)\left(\frac{1m}{3.281ft}\right)^2 = 0.1942 \cdot kg \cdot m^2$$

D: Power In = Power Out

0.1A @ 180V @ 1830rpm @ 0 ft-lb load

$$\omega = 1830rpm\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 191.6\frac{rad}{\text{sec}}$$

$$180V \cdot 0.1A = 18W = T \cdot \omega = (D \cdot \omega)\omega$$

$$D = 0.000490\frac{\text{Nm}}{\text{rad/sec}}$$

Plugging in numbers

$$\omega = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)$$

In Matlab

```
>> Kt = 0.9008;
>> J = 0.1942;
>> D = 0.000490;
>> R = 8.54;
>> L = 0.06059;
>> den = [J*L, J*R + D*L, D*R + Kt^2];
>> G = tf(Kt, [J*L, J*R + D*L, D*R + Kt^2]);
>> zpk(G)

      76.5558
--------------------
(s+140.5) (s+0.4935)

>>
```

Note: I think the data sheets are incorrect with respect to the rotor intertia.

- The motor weighs 23lb (10.43kg).
- The motor diameter is 4.50 inches (11.43cm).

Assume

- The rotor is 1/2 of the mass is the motor (probably a high estimate).
- The rotor is 1/2 the diameter of the motor (5.71cm diameter).
- The rotor is a solid cyllinder (for modeling)

The rotational inertia of a solid cyllinder (Wikipedia) is

$$J = \frac{1}{2} m r^2$$

$$J = \frac{1}{2} (5.216 kg)(0.02858 m)^2$$

$$J = 0.002130 \cdot kg \cdot m^2$$

which is 89x smaller than the data sheets. I think the data sheets are *wrong*

Using this inertia

```
>> Kt = 0.9008;
>> J = 0.00213;
>> D = 0.00049;
>> R = 8.54;
>> L = 0.06059;
>> den = [J*L, J*R + D*L, D*R + Kt^2];
>> G = tf(Kt, [J*L, J*R + D*L, D*R + Kt^2]);
>> zpk(G)

      6979.8778
---------------------
(s^2 + 141.2s + 6320)

>> eig(G)

 -70.5887 +36.5668i
 -70.5887 -36.5668i
```

This actually makes more sense. The poles are complex, meaning some overshoot. Motors are designed to drive something. Add inertia to the rotor and the poles will become real. (problem #5)

5) Assume this motor is used to power an electric bicycle at 20mph

- Motor speed @ 20mph = 1750 rpm
- Gear (wheel) used to convert 1750 rpm to 20mph
- Bicycle weight = 100kg

What is the gear reduction (wheel diameter) to convert 1750rpm to 20mph?

$$x = r\theta$$

$$\dot{x} = r\dot{\theta}$$

$$20mph\left(\frac{1609m}{mile}\right)\left(\frac{1hr}{3600\sec}\right) = 8.939\tfrac{m}{s}$$

$$1750rpm\left(\frac{2\pi\ rad}{rev}\right)\left(\frac{1\min}{60s}\right) = 183.3\tfrac{rad}{\sec}$$

$$8.939\tfrac{m}{s} = r \cdot 183.3\tfrac{rad}{\sec}$$

$$r = 0.04878m \quad \text{(48.78mm = 2 inch diameter)}$$

What is the inertia relative to the DC servo motor (bring the 100kg mass back to the motor through a gear)

The net inertia is

$$J = J_{motor} + r^2 M_{cart}$$

$$J = (0.00213 \cdot kg \cdot m^2) + (0.04878m)^2 \cdot 100kg$$

$$J = 0.2401 \cdot kg \cdot m^2$$

What is the transfer function (dynamics) for the bicycle / servo motor combination?

```
>> J = 0.2401;
>> G = tf(Kt, [J*L, J*R + D*L, D*R + Kt^2]);
>> zpk(G)

      61.9206
--------------------
(s+140.6) (s+0.3988)
```

The extra inertia made the complex pole real like you'd expect. If used to drive an electric bike, this motor would get you up to speed in 10 seconds (2% settling time).