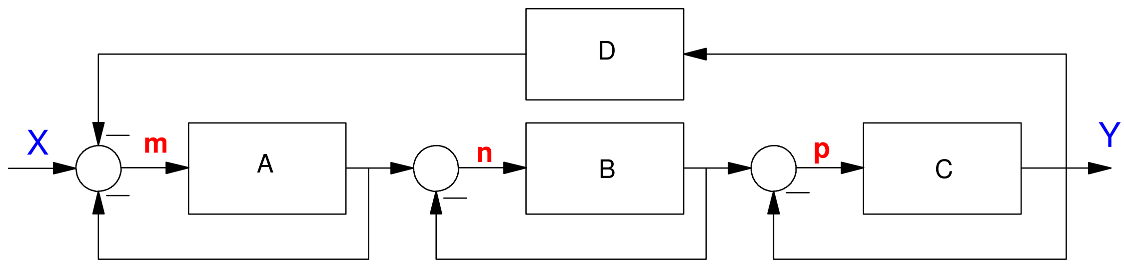
Homework #5: ECE 461/661

Block Diagrams, Canonical Forms, Electrical Circuits. Due Monday, September 19th

Block Diagrams

1) Determine the transfer function from X to Y



Shortcut (not quite right)

$$Y = \left(\frac{ABC}{1+A+B+C+ABCD}\right)X$$

Shortcut (right)

$$Y = \left(\frac{\left(\frac{A}{1+A}\right)\left(\frac{B}{1+B}\right)\left(\frac{C}{1+C}\right)}{1+\left(\frac{A}{1+A}\right)\left(\frac{B}{1+B}\right)\left(\frac{C}{1+C}\right)D}\right)X$$

$$Y = \left(\frac{ABC}{(1+A)(1+B)(1+C)+ABCD}\right)X$$

Long Way

$$m = X - DCp - Am$$

$$n = Am - Bn$$

$$p = Bn - Cp$$

$$Y = Cp$$

Simplify

$$m = \left(\frac{1}{1+A}\right)X + \left(\frac{DC}{1+A}\right)p$$

$$n = \left(\frac{A}{1+B}\right)m$$

$$p = \left(\frac{B}{1+C}\right)p$$

$$p = \left(\frac{B}{1+C}\right)\left(\frac{A}{1+B}\right)\left(\left(\frac{1}{1+A}\right)X + \left(\frac{DC}{1+A}\right)p\right)$$

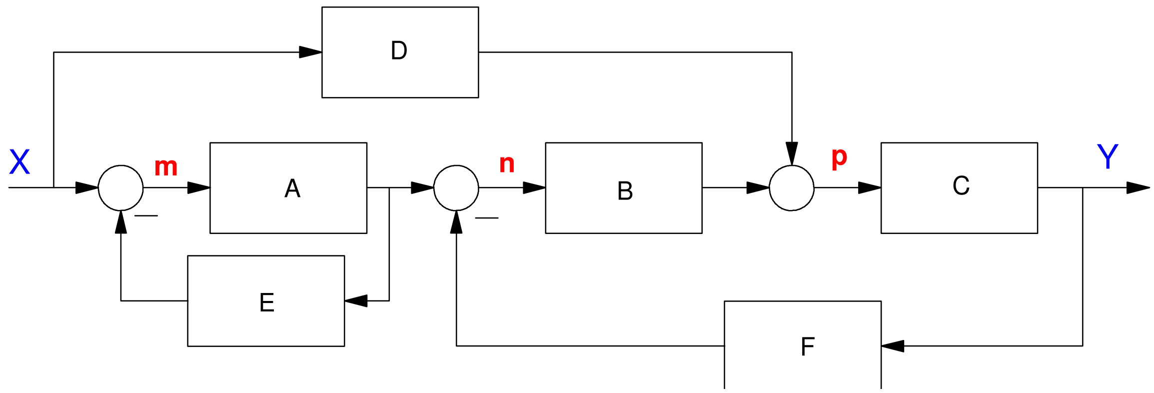
$$(1+A)(1+B)(1+C)p = ABX + BADCp$$

$$p = \left(\frac{AB}{(1+A)(1+B)(1+C)+ABCD}\right)X$$

$$\boxed{Y = \left(\frac{ABC}{(1+A)(1+B)(1+C)+ABCD}\right)X}$$

which matches

2) Determine the transfer funciton from X to Y



Shortcut: This is actually two systems cascaded

$$Y = \left(\frac{A}{1+AE}\right)\left(\frac{BC}{1+BCF}\right)X + \left(\frac{DC}{1+BCF}\right)X$$

Long Way

$$m = X - EAm$$

$$n = Am - FCp$$

$$p = Bn + DX$$

$$Y = Cp$$

Simplify

$$m = \left(\frac{1}{1+EA}\right)X$$

$$n = Am - FC(Bn + DX)$$

$$(1 + FCB)n = A\left(\frac{1}{1+EA}\right)X - FCDX$$

$$(1 + FCB)(1 + EA)n = AX - (1 + EA)FCDX$$

$$(1 + FCB)(1 + EA)n = (A - FCD - EAFCD)X$$

$$n = \left(\frac{A-CDF-ACDEF}{(1+FCB)(1+EA)}\right)X$$

$$p = Bn + DX = B\left(\frac{A-CDF-ACDEF}{(1+FCB)(1+EA)}\right)X + DX$$

$$Y = Cp = CB\left(\frac{A-CDF-ACDEF}{(1+FCB)(1+EA)}\right)X + CDX$$

$$(1 + FCB)(1 + EA)Y = (CBA - CBCDF - CBACDEF + CD(1 + FCB)(1 + EA))X$$

$$= (CBA - CBCDF - CBACDEF + CD + CDFCB + CDEA + CDFCBEA)X$$

$$(1 + FCB)(1 + EA)Y = (CBA + CD + CDEA)X$$

$$Y = \left(\frac{CBA+CD+CDEA}{(1+FCB)(1+EA)}\right)X$$

$$Y = \left(\frac{CBA+CD(1+EA)}{(1+FCB)(1+EA)}\right)X$$

$$Y = \left(\left(\frac{CBA}{(1+FCB)(1+EA)}\right) + \left(\frac{CD}{(1+FCB)}\right)\right)X$$

which matches

Canonical Forms

3) Give two different state-space models that produce the following transfer function

$$Y = \left(\frac{30s+30}{(s+2)(s+3)(s+4)}\right)U$$

Multiply out and use controller canonical form

$$Y = \left(\frac{30s+30}{s^3+9s^2+26s+24}\right)U$$

$$sX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 30 & 30 & 0 \end{bmatrix} X$$

Transpose to get obserer canonical form

$$sX = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix} X + \begin{bmatrix} 30 \\ 30 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} X$$

Do partial fractions to get Jordan form

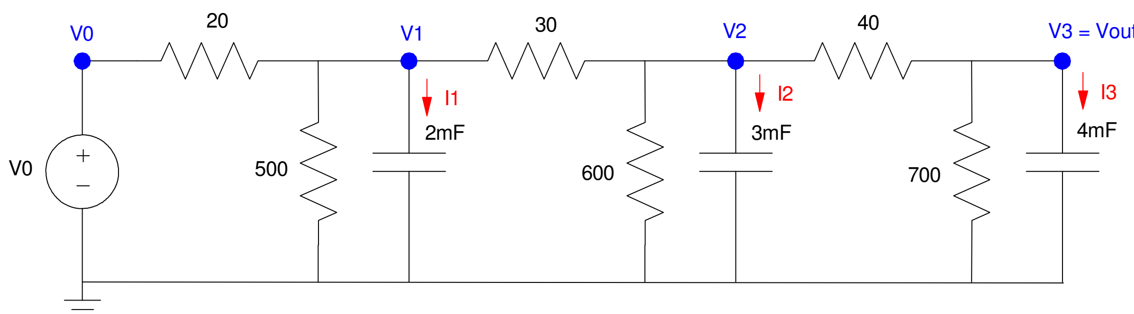
$$Y = \left(\left(\frac{-15}{s+2}\right) + \left(\frac{60}{s+3}\right) + \left(\frac{-45}{s+4}\right)\right)U$$

$$sX = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} -15 & 60 & -45 \end{bmatrix} X$$

Electrical Ciruits

4) Using state-space methods, find the transfer function from V0 to V3



$$I_1 = 0.002\dot{V}_1 = \left(\frac{V_0 - V_1}{20}\right) + \left(\frac{-V_1}{500}\right) + \left(\frac{V_2 - V_1}{30}\right)$$

$$I_2 = 0.003\dot{V}_2 = \left(\frac{V_1 - V_2}{30}\right) + \left(\frac{-V_2}{600}\right) + \left(\frac{V_3 - V_2}{40}\right)$$

$$I_3 = 0.004\dot{V}_3 = \left(\frac{V_2 - V_3}{40}\right) + \left(\frac{-V_3}{700}\right)$$

Simplify

$$\dot{V}_1 = 25V_0 - 42.667V_1 + 16.667V_2$$

$$\dot{V}_2 = 11.111V_1 - 20V_2 + 8.333V_3$$

$$\dot{V}_3 = 6.25V_2 - 6.607V_3$$

Place in matrix (state-space) form

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -42.667 & 16.667 & 0 \\ 11.111 & -20 & 8.333 \\ 0 & 6.25 & -6.607 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = V_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Place in Matlab and find the transfer funciton

```
>> A = [-42.667,16.667,0 ; 11.111,-20,8.333 ; 0,6.25,-6.607]

A =

  -42.6670   16.6670         0
   11.1110  -20.0000    8.3330
         0    6.2500   -6.6070

>> B = [25;0;0];
>> C = [0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

          1736.0938
----------------------------
(s+49.27) (s+17.46) (s+2.549)
```

5) Using state-space methods, find the transfer function from V0 to V2

Just change the C matrix

$$Y = V_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

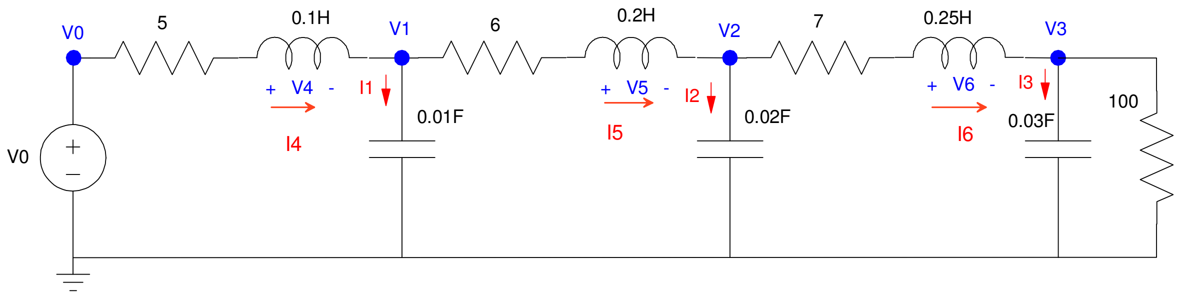
```
>> C = [0,1,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

      277.775 (s+6.607)
----------------------------
(s+49.27) (s+17.46) (s+2.549)
```

Note that when you change the output

- The zeros change, but
- The poles are unchanged

6) Express the dynamics for the following RLC circuit in state-space form.

- Find the transfr function from V0 to V3



$$I_1 = 0.01sV_1 = I_4 - I_5$$

$$I_2 = 0.02sV_2 = I_5 - I_6$$

$$I_3 = 0.03sV_3 = I_6 - \left(\frac{V_3}{100}\right)$$

$$V_4 = 0.1sI_4 = V_0 - 5I_4 - V_1$$

$$V_5 = 0.2sI_5 = V_1 - 6I_5 - V_2$$

$$V_6 = 0.25sI_6 = V_2 - 7I_6 - V_3$$

Solving for the derivative of the states

$$sV_1 = 100I_4 - 100I_5$$

$$sV_2 = 50I_5 - 50I_6$$

$$sV_3 = 33.33I_6 - 0.333V_3$$

$$sI_4 = 10V_0 - 50I_4 - 10V_1$$

$$sI_5 = 5V_1 - 30I_5 - 5V_2$$

$$sI_6 = 4V_2 - 28I_6 - 4V_3$$

Place in matrix form

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \\ sI_4 \\ sI_5 \\ sI_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 100 & -100 & 0 \\ 0 & 0 & 0 & 0 & 50 & -50 \\ 0 & 0 & -0.333 & 0 & 0 & 33.33 \\ -10 & 0 & 0 & -50 & 0 & 0 \\ 5 & -5 & 0 & 0 & -30 & 0 \\ 0 & 4 & -4 & 0 & 0 & -28 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 0 \end{bmatrix} V_0$$

Solve for the transfer function to V3 using Matlab

```
>> a1 = [0,0,0,100,-100,0];
>> a2 = [0,0,0,0,50,-50];
>> a3 = [0,0,-0.333,0,0,33.33];
>> a4 = [-10,0,0,-50,0,0];
>> a5 = [5,-5,0,0,-30,0];
>> a6 = [0,4,-4,0,0,-28];
>> A = [a1;a2;a3;a4;a5;a6]

A =

         0         0         0  100.0000 -100.0000         0
         0         0         0         0   50.0000  -50.0000
         0         0   -0.3330         0         0   33.3300
  -10.0000         0         0  -50.0000         0         0
    5.0000   -5.0000         0         0  -30.0000         0
         0    4.0000   -4.0000         0         0  -28.0000

>> eig(A)

ans =

 -19.1447 +33.5430i
 -19.1447 -33.5430i
 -37.3121
 -15.5065 +13.0642i
 -15.5065 -13.0642i
  -1.7186

>> B = [0;0;0;10;0;0];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


                            33330000
----------------------------------------------------------------
(s+37.31) (s+1.719) (s^2 + 31.01s + 411.1) (s^2 + 38.29s + 1492)

>>
```

7) Assume V0 = 0. Specify the initial conditions so that the total energy at t = 0 is 1.0 Joules and

- The transients decay as slow as possible
- The transients decay as fast as possible

This is an eigenvector problem. The eigenvalues and eigenvectors are:

```
>> [M,V] = eig(A)

M (eigenvector)

  -0.9292              -0.9292              -0.7116
   0.2417 - 0.0835i     0.2417 + 0.0835i    -0.2797
  -0.0231 + 0.0163i    -0.0231 - 0.0163i    -0.0781
   0.1380 - 0.1501i     0.1380 + 0.1501i     0.5609
  -0.0399 + 0.1616i    -0.0399 - 0.1616i     0.2953
  -0.0033 - 0.0325i    -0.0033 + 0.0325i     0.0866

V (eigenvalue)

 -19.1447 -33.5430i -19.1447 +33.5430i -37.3121
```

```
M (eigenvector)

   0.5231 - 0.1566i     0.5231 + 0.1566i    -0.2718
   0.6811               0.6811              -0.5637
  -0.3908 + 0.0603i    -0.3908 - 0.0603i    -0.7756
  -0.1176 + 0.0899i    -0.1176 - 0.0899i     0.0563
  -0.0569 - 0.0027i    -0.0569 + 0.0027i     0.0516
   0.1543 - 0.1806i     0.1543 + 0.1806i     0.0322

V (eigenvalues)

 -15.5065 -13.0642i -15.5065 +13.0642i  -1.7186
```

The fast mode (blue) decays as exp(-37.31t). Using these values, the energy in the initial condition is

$$E = \Sigma \tfrac{1}{2} C V^2 + \Sigma \tfrac{1}{2} L I^2$$

Using this initial condition, the initial energy is 290mJ

```
X0 = M(:,3);

>> LC = [0.01,0.02,0.03,0.1,0.2,0.3]

LC =     0.0100     0.0200     0.0300     0.1000     0.2000     0.3000

>> J = 0.5 * LC * (X0 .^ 2)

J =    0.0290
```

Scale X0 so that the initial energy is 1.00J

```
>> X0 / sqrt(J)

V1   -4.1801
V2   -1.6430
V3   -0.4586
I4    3.2945
I5    1.7348
I6    0.5088
```

Slow Mode

```
>> X0 = M(:,6);
>> J = 0.5 * LC * (X0 .^ 2)

J =    0.0132

>> X0 / sqrt(J)

V1   -2.3699
V2   -4.9159
V3   -6.7631
I4    0.4908
I5    0.4501
I6    0.2812

>>
```