# Homework #10: ECE 461/661

z-Transforms, s to z conversion, Root Locus in the z-Domain.    Due Monday, November 7th

## z-Transforms

1) Determine the difference equation that relates X and Y

$$Y = \left(\frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)}\right)X$$

Cross multiply

$$((z-0.9)(z-0.8)(z-0.5))Y = 0.5zX$$

$$(z^3 - 2.2z^2 + 1.57z - 0.36)Y = 0.5zX$$

which means

$$\boxed{y(k+3) - 2.2y(k+2) + 1.57y(k+1) - 0.36y(k) = 0.5x(k+1)}$$

If you don't like using future values, do a change in variable

k+3 = m (or k)

$$\boxed{y(k) - 2.2y(k-1) + 1.57y(k-2) - 0.36y(k-3) = 0.5x(k-2)}$$

Both answers are correct

2) Determine y(k) assuming

$$Y = \left(\frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)}\right)X \qquad x(t) = 5\cos(2t) + 2\sin(2t)$$

$$T = 0.01$$

Use phasors

$$s = j2$$

$$z = e^{sT} = e^{j0.02} = 1\angle 0.02 rad$$

$$X = 5 - j2$$

$$Y = \left(\frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)}\right)X$$

$$Y = \left(\frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)}\right)_{z=e^{j0.02}} \cdot (5 - j2)$$

$$Y = 20.1780 - j16.9277$$

which means

$$\boxed{y(t) = 20.1780\cos(2t) + 16.9277\sin(2t)}$$

If you prefer using k

$$t = kT = 0.01k$$

$$\boxed{y(k) = 20.1780\cos(0.02k) + 16.9277\sin(0.02k)}$$

Both answers are correct

3) Determine y(k) assuming

$$Y = \left( \frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)} \right) X \qquad x(k) = u(k)$$

Substitute the z-transform of X

$$Y = \left( \frac{0.05z}{(z-0.9)(z-0.8)(z-0.5)} \right) \left( \frac{z}{z-1} \right)$$

Factor out a z

$$Y = \left( \frac{0.05z}{(z-1)(z-0.9)(z-0.8)(z-0.5)} \right) z$$

Do a partial fraction expansion

$$Y = \left( \left( \frac{5}{z-1} \right) + \left( \frac{-11.25}{z-0.9} \right) + \left( \frac{6.667}{z-0.8} \right) + \left( \frac{-0.4167}{z-0.5} \right) \right) z$$

Multiply through by z

$$Y = \left( \frac{5z}{z-1} \right) + \left( \frac{-11.25z}{z-0.9} \right) + \left( \frac{6.667z}{z-0.8} \right) + \left( \frac{-0.4167z}{z-0.5} \right)$$

and take the inverse z-transform

$$y(k) = \left( 5 - 11.25(0.9)^k + 6.667(0.8)^k - 0.4167(0.5)^k \right) u(k)$$

## s to z conversion

3) Determine the discrete-time equivalent of G(s). Assume T = 0.1 seconds

$$G(s) = \left(\frac{903}{(s+0.81)(s+5.20)(s+13.59)(s+25.25)}\right)$$

In matlab

```
>> s = [-0.81,-5.20,-13.59,-25.25]'

   -0.8100
   -5.2000
  -13.5900
  -25.2500

>> T = 0.1;
>> z = exp(s*T)

    0.9222
    0.5945
    0.2569
    0.0801

>> DC = 903 / prod(s)

DC =    0.6248

>> k = DC * prod(1-z)

k =    0.0135
```

$$G_z(z) = \left(\frac{0.0135}{(z-0.9222)(z-0.5945)(z-0.2569)(z-0.0801)}\right)$$

To match the phase (pick s = j0.1), add a couple zeros at z = 0.

$$G_s(j0.1) = 0.6199\angle -8.7822^0$$

$$G_z(z = e^{sT}) = 0.6199\angle -10.1351^0$$

so the digital filter has too much delay. Each zero adds 0.573 degrees. The numbers of zeros you need is 2.35

$$n = \left(\frac{\angle G(s)-\angle G(z)}{\angle z}\right) = \left(\frac{-8.7822^0-\left(-10.1351^0\right)}{0.573^0}\right) = 2.35$$

n has to be an integer so round down to 2

$$G_z(z) \approx \left(\frac{0.0135z^2}{(z-0.9222)(z-0.5945)(z-0.2569)(z-0.0801)}\right)$$

4) Determine the discrete-time equivalent of G(s). Assume T = 0.01 seconds

```
>> s = [-0.81,-5.20,-13.59,-25.25]'

   -0.8100
   -5.2000
  -13.5900
  -25.2500

>> T = 0.01;
>> z = exp(s*T)

    0.9919
    0.9493
    0.8729
    0.7769

>> DC = 903 / prod(s)

DC =    0.6248

>> k = DC * prod(1-z)

k =  7.2416e-006
```

$$G(z) \approx \left( \frac{7.2416 \cdot 10^{-6}}{(z-0.9919)(z-0.9493)(z-0.8729)(z-0.7769)} \right)$$

Add zeros at z=0 to match the phase somewhere, like s = j0.1

$$G_s(s = j0.1) = 0.6199\angle -8.7882^0$$

$$G_z(z = e^{sT}) = 0.6199\angle -8.9049^0$$

$$z = e^{sT} = 1\angle 0.0573^0$$

$$n = \left( \frac{-8.7882^0 - \left(-8.9049^0\right)}{0.0573^0} \right) = 2.0373$$

or in Matlab (s, z, and k are defined above)

```
>> Gs = zpk([],s,903);
>> Gz = zpk([],z,k);
>> s = j*0.1;
>> z = exp(s*T);
>> Gs1 = evalfr(Gs, s);
>> Gz1 = evalfr(Gz, z);
>> n = ( angle(Gs1) - angle(Gz1) ) / angle(z)

n =    2.0373
```

Round to the nearest integer

```
n = 2
```

$$G(z) \approx \left( \frac{7.2416 \cdot 10^{-6} \cdot z^2}{(z-0.9919)(z-0.9493)(z-0.8729)(z-0.7769)} \right)$$

## Root Locus in the z-Domain

Assume T = 0.1 seconds.

$$G(s) = \left(\frac{903}{(s+0.81)(s+5.20)(s+13.59)(s+25.25)}\right)$$

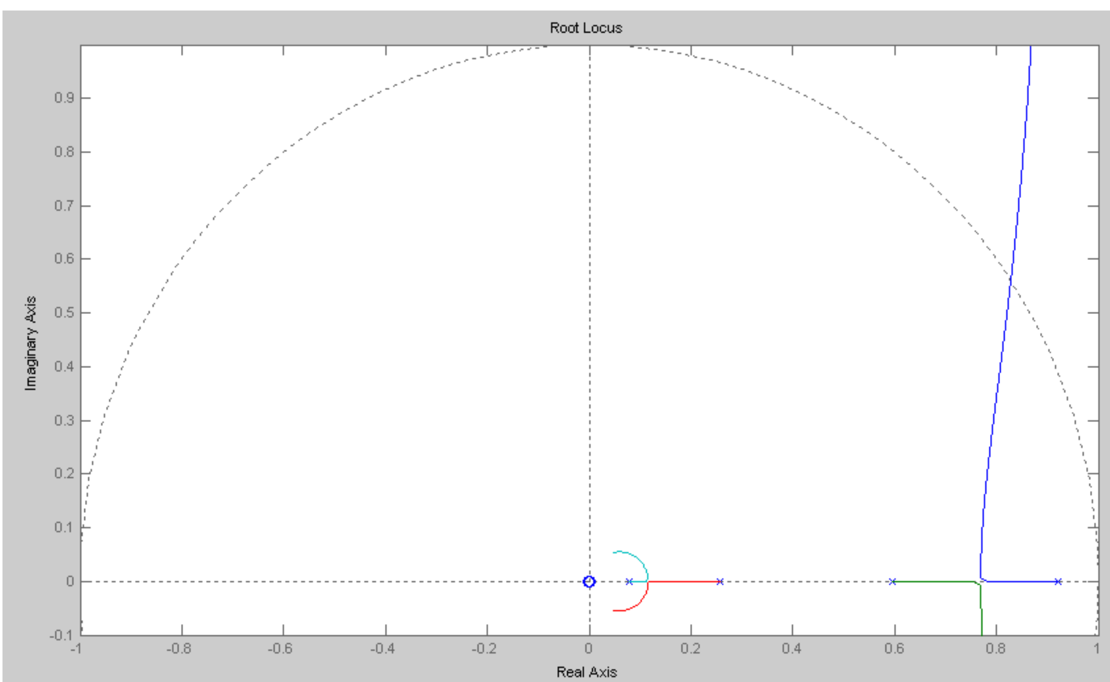
5) Draw the root locus for G(z)

From problem #3

$$G_z(z) \approx \left(\frac{0.0135z^2}{(z-0.9222)(z-0.5945)(z-0.2569)(z-0.0801)}\right)$$

In Matlab

```
s = [-0.81,-5.20,-13.59,-25.25]';
T = 0.1;
z = exp(s*T);
k = 0.0135;
>> Gz = zpk([0,0],z,k,T)

                0.013474 z^2
--------------------------------------------
(z-0.9222) (z-0.5945) (z-0.2569) (z-0.08006)

Sampling time (seconds): 0.1

>> k = logspace(-2,2,1000)';
>> rlocus(Gz,k);
```



6) Find k for no overshoot in the step response

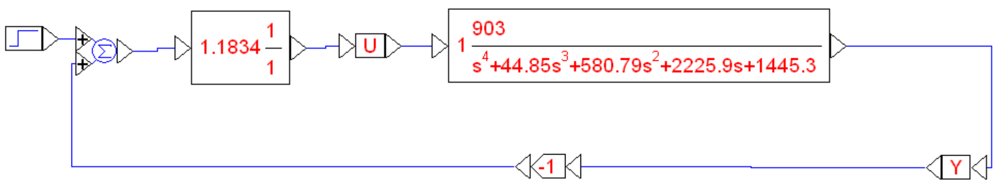
- Simulate the closed-loop system's step response

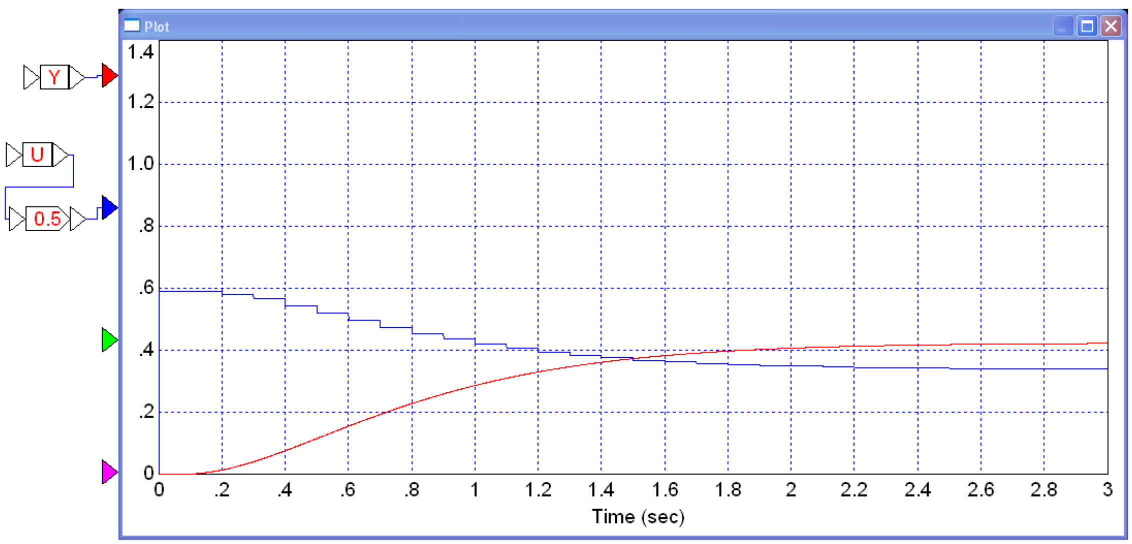
Find the spot on the root locus.

- Pick the breakaway point

```
>> z = 0.77;
>> evalfr(Gz,z)

ans =    -0.8450

>> k = 1/abs(ans)
```

**k =     1.1834**



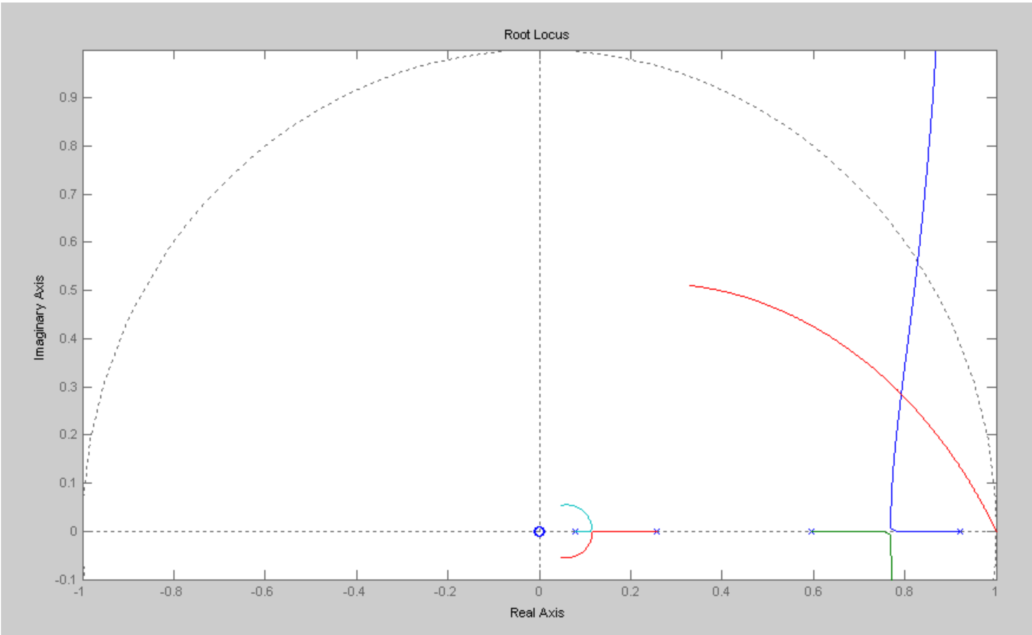



7) Find k for 20% overshoot for a step response (damping ratio = 0.4559)

- Simulate the closed-loop system's step response

Add the damping line to the previous root locus

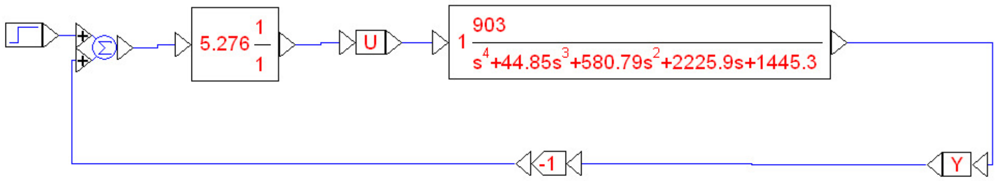
```
>> hold on
>> s = [0:0.01:5]' * (-1 + j*2);
>> z = exp(s*T);
>> plot(real(z),imag(z),'r')
```

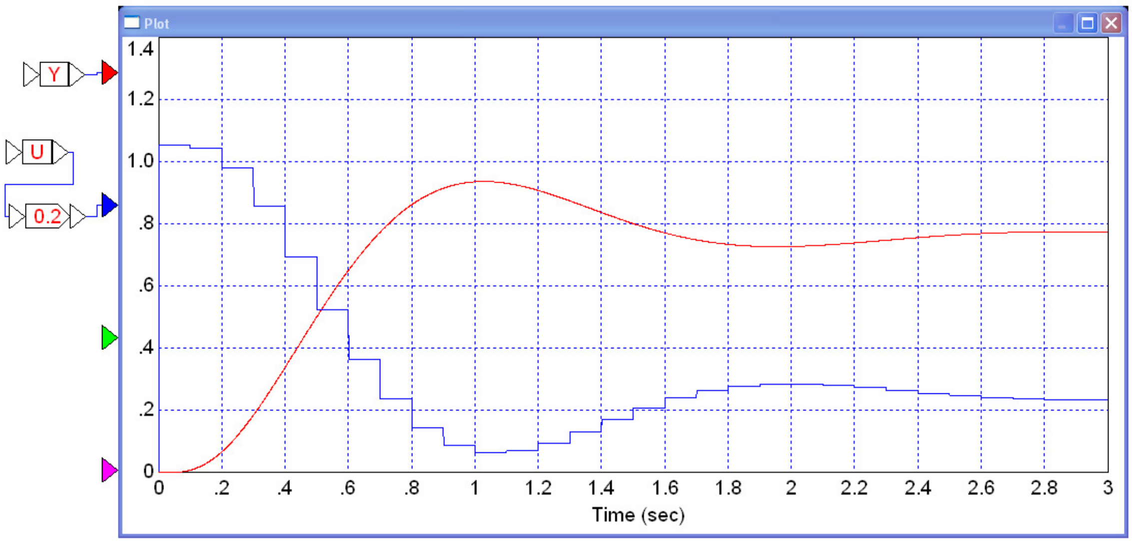



Find where the root locus hits the damping line

```
>> evalfr(Gz,z)

ans = -1.8973e-001 +4.3742e-005i

>> k = 1/abs(ans)

k =  5.2706e+000
```

Check your response

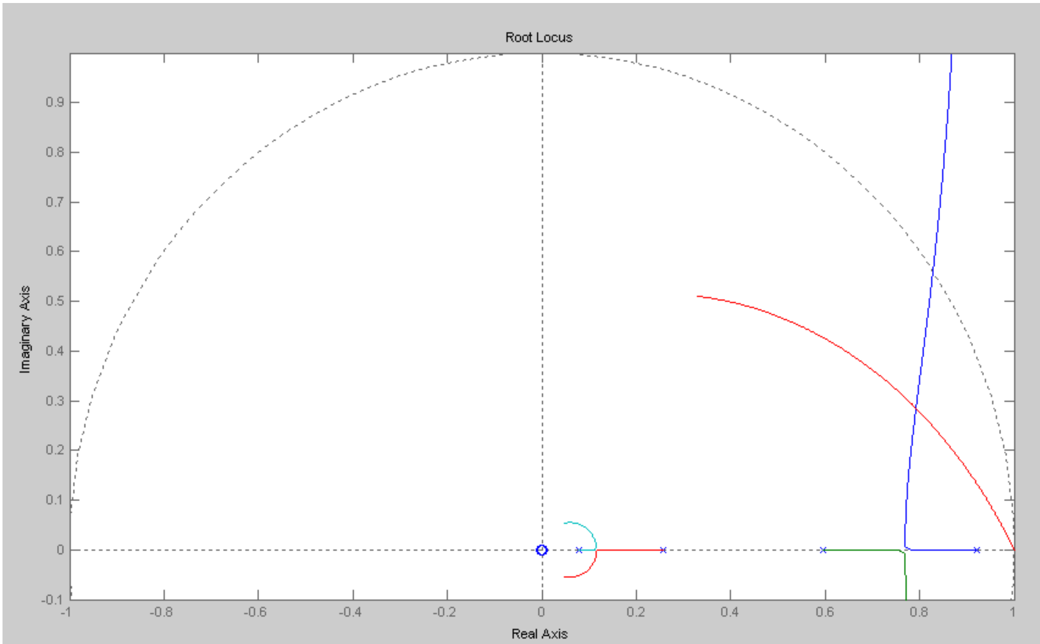

5.276
1
1
U
1
903
s^4+44.85s^3+580.79s^2+2225.9s+1445.3
-1
Y


Y
U
0.2
Plot
1.4
1.2
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
Time (sec)

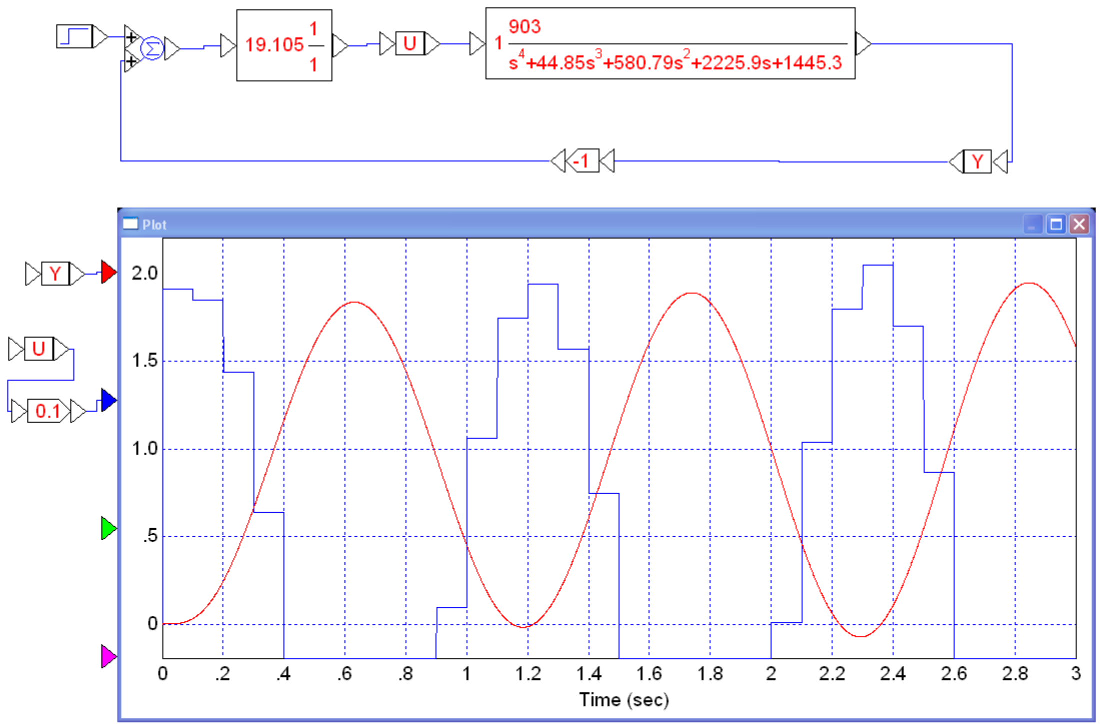
8) Find k for a damping ratio of 0.00

- Simulate the closed-loop system's step response

Find the spot where the root locus intersects the unit circle




```
>> z = 0.8281 + j*0.5597;
>> evalfr(Gz,z)

ans = -5.2342e-002 +4.8699e-006i

>> k = 1/abs(ans)

k =  1.9105e+001
```




Note:

- Root locus works in the s-domain
- Root locus workin the z-domain