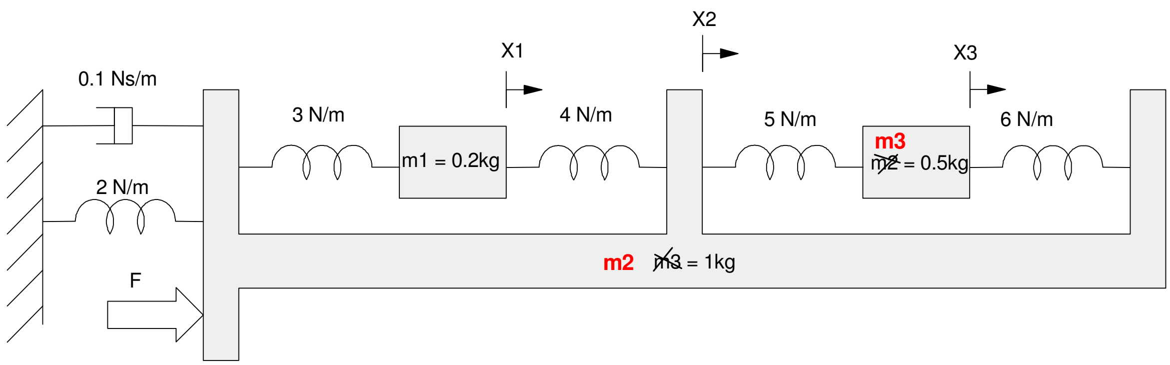
Homework #6: ECE 461/661

Mass-Spring Systems, Rotational Systems, DC Motors.   Due Monday, September 25th

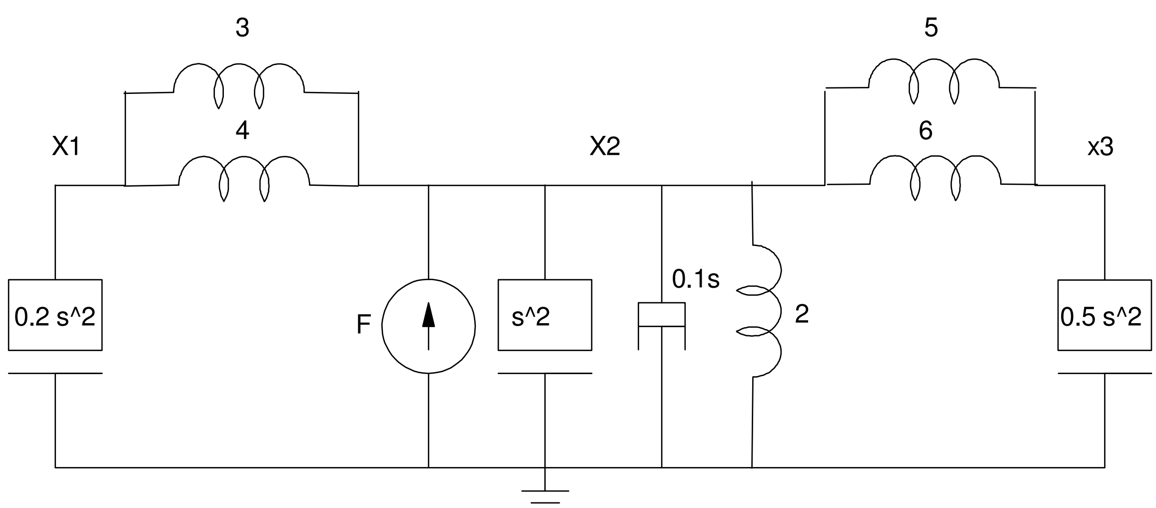
Mass Spring systems

1) (20pt) Draw the circuit equivalent for the following mass-spring systems.

- Express the dynamics in state-space form
- Find the transfer function from F to X2
- Plot the step response from F to X2



Step 1: Draw the circuit equivalent



Step 2: Write the voltage node equations

$$(0.2s^2 + 3 + 4)x_1 - (3 + 4)x_2 = 0$$

$$(s^2 + 0.1s + 2 + 3 + 4 + 5 + 6)x_2 - (3 + 4)x_1 - (5 + 6)x_3 = F$$

$$(0.5s^2 + 5 + 6)x_3 - (5 + 6)x_2 = 0$$

Step 3: Solving for the highest derivative

$$s^2 x_1 = -35x_1 + 35x_2$$

$$s^2 x_2 = -0.1 s x_2 - 20x_2 + 7x_1 + 11x_3 + F$$

$$s^2 x_3 = -22x_3 + 22x_2$$

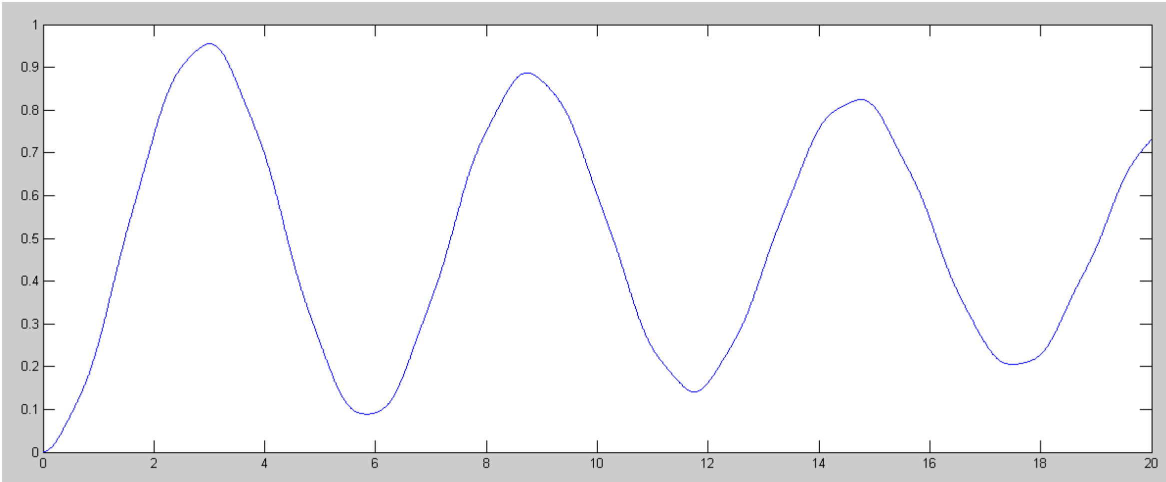
Place in matrix form (state-space form)

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -35 & 35 & 0 & 0 & 0 & 0 \\ 7 & -20 & 11 & 0 & -0.1 & 0 \\ 0 & 22 & -22 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} F$$

Throw into Matlab and find the transfr funciton to X2

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-35,35,0;7,-20,11;0,22,-22];
>> B = [0,0,0;0,-0.1,0;0,0,0];
>> A = [Z,I ; K,B]
         0          0          0    1.0000          0          0
         0          0          0         0     1.0000          0
         0          0          0         0          0     1.0000
  -35.0000    35.0000          0         0          0          0
    7.0000   -20.0000    11.0000         0    -0.1000          0
         0    22.0000   -22.0000         0          0          0

>> eig(A)

  -0.0179 + 6.9319i
  -0.0179 - 6.9319i
  -0.0039 + 5.2718i
  -0.0039 - 5.2718i
  -0.0282 + 1.0735i
  -0.0282 - 1.0735i

>> B = [0;0;0;0;1;0];
>> C = [0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                      (s^2 + 22) (s^2 + 35)
---------------------------------------------------------------------------
(s^2 + 0.05648s + 1.153) (s^2 + 0.007735s + 27.79) (s^2 + 0.03579s + 48.05)
```

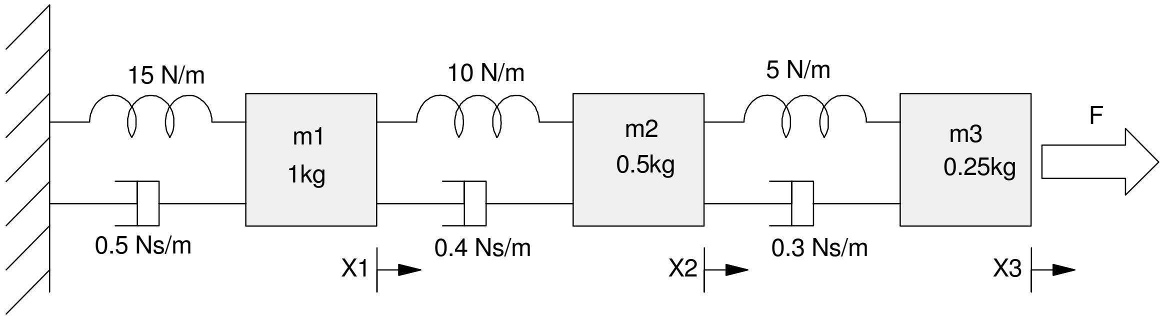
```
>> t = [0:0.01:20]';
>> y = step(G,t);
```



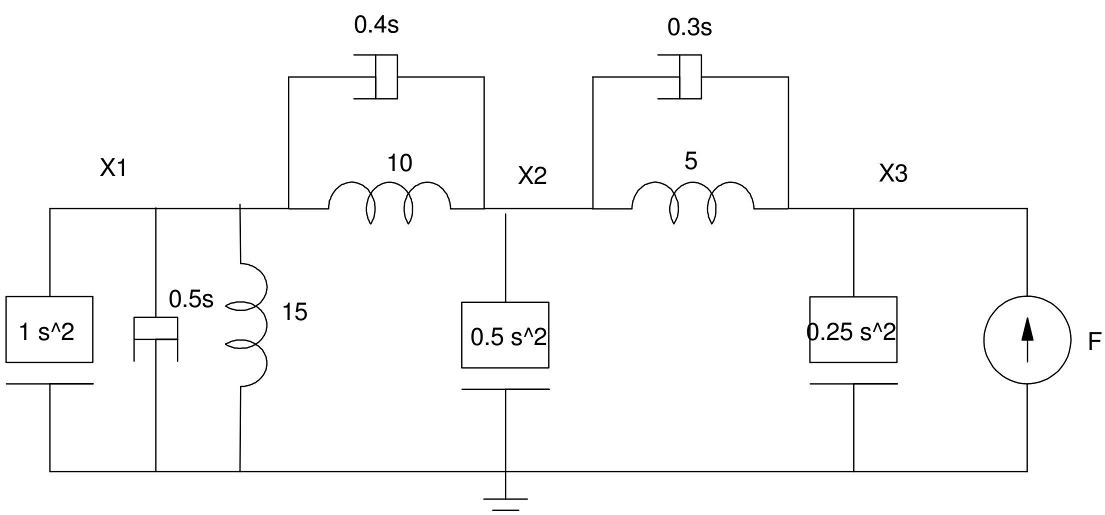
2) (20pt) Draw the circuit equivalent for the following mass-spring systems.

- Express the dynamics in state-space form
- Find the transfer function from F to X3

Plot the step response from F to X3



Draw the circuit equivalent



Write the voltage node equations

$$(s^2 + 0.5s + 15 + 0.4s + 10)x_1 - (0.4s + 10)x_2 = 0$$

$$(0.5s^2 + 10 + 0.4s + 5 + 0.3s)x_2 - (0.4s + 10)x_1 - (0.3s + 5)x_3 = 0$$

$$(0.25s^2 + 0.3s + 5)x_3 - (0.3s + 5)x_2 = F$$

Solve for the highest derivative

$$s^2 x_1 = (-0.9s - 25)x_1 + (0.4s + 10)x_2$$

$$s^2 x_2 = (0.8s + 20)x_1 + (-1.4s - 30)x_2 + (0.6s + 10)x_3$$

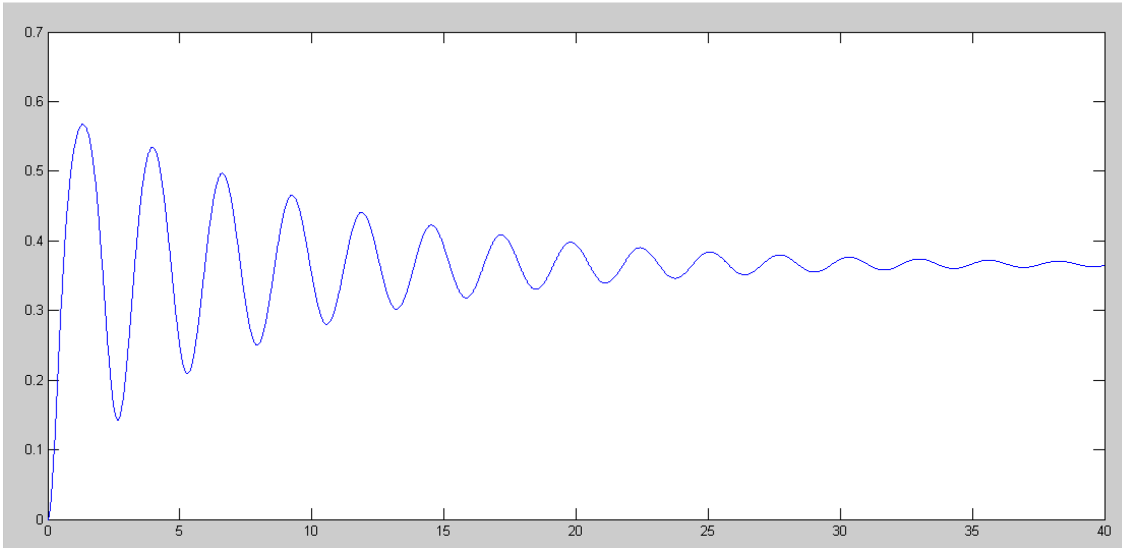
$$s^2 x_3 = (1.2s + 20)x_2 + (-1.2s - 20)x_3 + 4F$$

Place in matrix form

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -25 & 10 & 0 & -0.9 & 0.4 & 0 \\ 20 & -30 & 10 & 0.8 & -1.4 & 0.6 \\ 0 & 20 & -20 & 0 & 1.2 & -1.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} F$$

Throw into Matlab

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-25,10,0;20,-30,10;0,20,-20];
>> B = [-0.9,0.4,0;0.8,-1.4,0.6;0,1.2,-1.2];
>> A = [Z,I ; K,B]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
  -25.0000    10.0000          0    -0.9000     0.4000          0
   20.0000   -30.0000    10.0000     0.8000    -1.4000     0.6000
         0    20.0000   -20.0000          0     1.2000    -1.2000

>> B = [0;0;0;0;0;4];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> eig(A)

  -1.1009 + 6.7295i
  -1.1009 - 6.7295i
  -0.5399 + 4.7342i
  -0.5399 - 4.7342i
  -0.1092 + 2.3815i
  -0.1092 - 2.3815i

>> zpk(G)


         4 (s^2 + 0.5494s + 13.15) (s^2 + 1.751s + 41.83)
-----------------------------------------------------------------

(s^2 + 0.2184s + 5.683) (s^2 + 1.08s + 22.7) (s^2 + 2.202s + 46.5)


>> t = [0:0.01:40]';
>> y = step(G,t);
>> evalfr(G,0)


    0.3667

>> plot(t,y)
>>
```

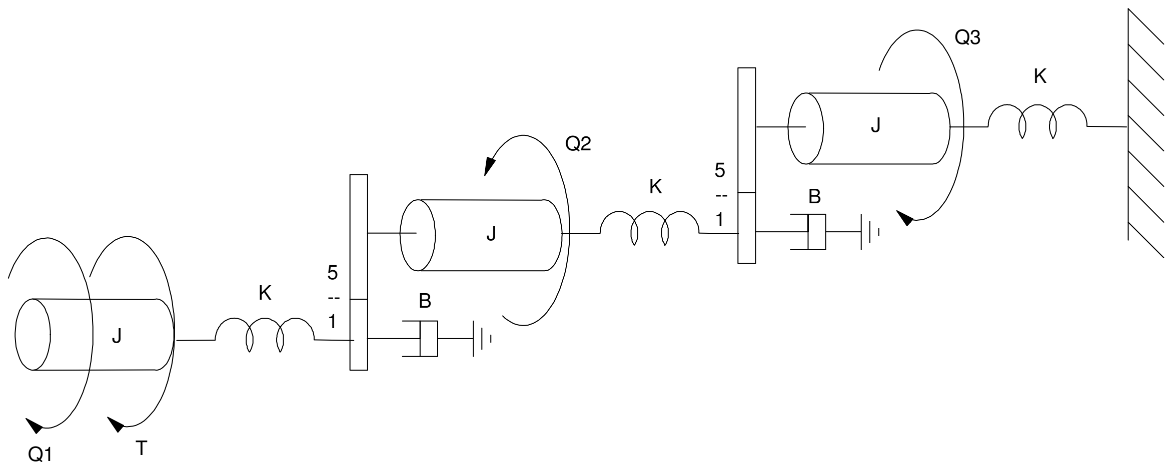


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35
40

Rotational Systems

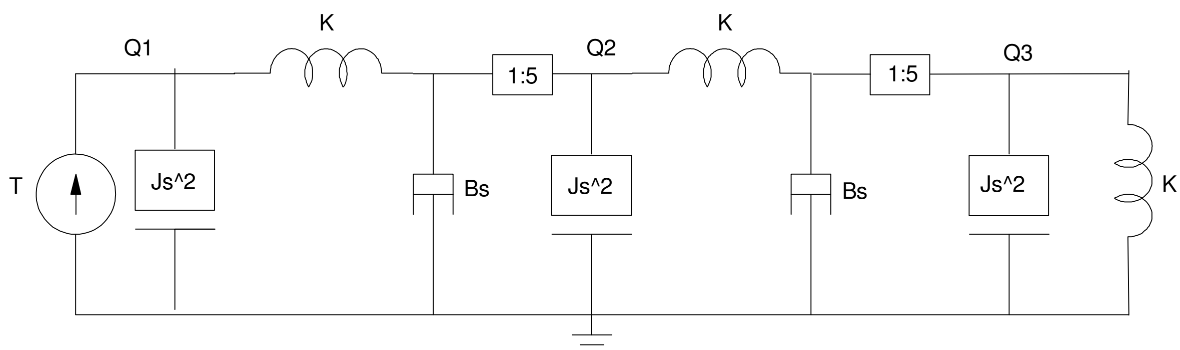
3) Draw the circuit equivalent for the following rotational system.

- Express the dynamics in state-space form
- Find the transfer function from T to Q1
- Plot the step response from T to Q1

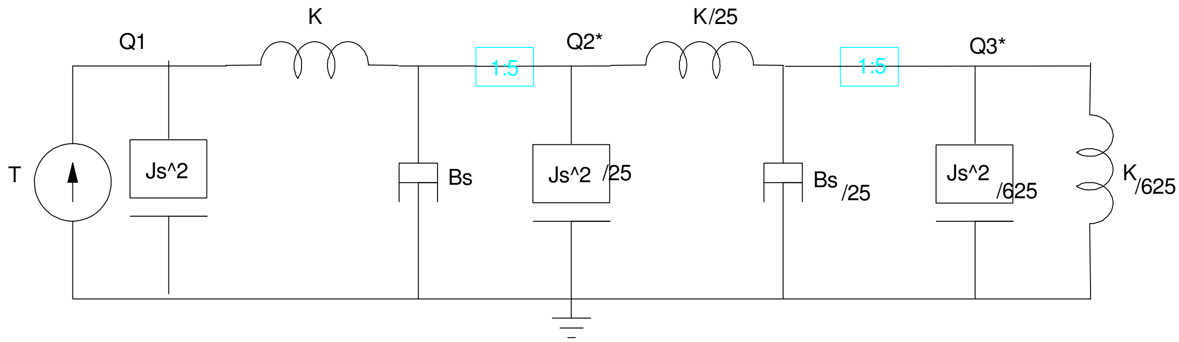


Problem 3: J = 0.2 Kg m / s^2, B = 0.1 Ns/m, K = 5 Nm/rad

Start with the circuit equivalent



Remove the gears



Write the node equations

$$(Js^2 + K)\theta_1 - (K)\theta_2 = T$$

$$\left(\frac{J}{25}s^2 + Bs + K + \frac{K}{25}\right)\theta_2 - (K)\theta_1 - \left(\frac{K}{25}\right)\theta_3 = 0$$

$$\left(\frac{J}{625}s^2 + \frac{B}{25}s + \frac{K}{25} + \frac{K}{625}\right)\theta_3 - \left(\frac{K}{25}\right)\theta_2 = 0$$

Plug in numbers and solve for the highest derivative

- J = 0.2 Kg m / s^2, B = 0.1 Ns/m, K = 5 Nm/rad

$$s^2\theta_1 = (-25)\theta_1 + (25)\theta_2 + 5T$$

$$s^2\theta_2 = (625)\theta_1 + (-12.5s - 650)\theta_2 + (25)\theta_3$$

$$s^2\theta_3 = (625)\theta_2 + (-12.5s - 650)\theta_3$$

Place in state-space form

$$s\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ s\theta_1\\ s\theta_2\\ s\theta_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -25 & 25 & 0 & 0 & 0 & 0\\ 625 & -650 & 25 & 0 & -12.5 & 0\\ 0 & 625 & -650 & 0 & 0 & -12.5\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ s\theta_1\\ s\theta_2\\ s\theta_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 5\\ 0\\ 0\end{bmatrix}F$$

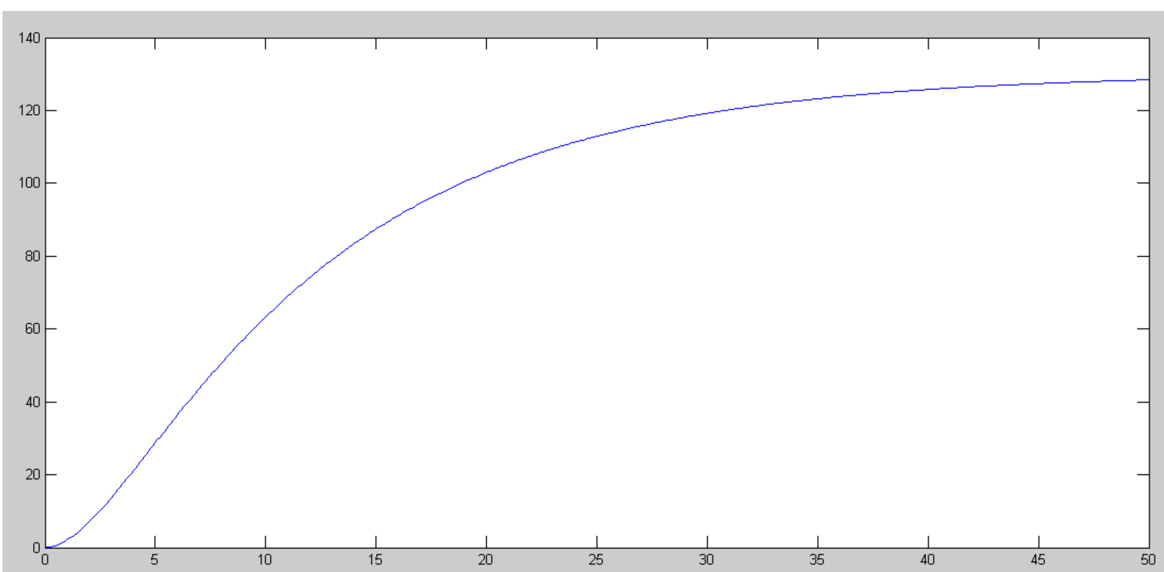
Solve in Matlab

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-25,25,0;625,-650,25;0,625,-650];
>> B = [0,0,0;0,-12.5,0;0,0,-12.5];
>> A = [Z,I ; K,B]
```

```
        0         0         0    1.0000         0         0
        0         0         0         0    1.0000         0
        0         0         0         0         0    1.0000
 -25.0000   25.0000         0         0         0         0
 625.0000 -650.0000   25.0000         0  -12.5000         0
        0  625.0000 -650.0000         0         0  -12.5000

>> B = [0;0;0;5;0;0];
>> C = [1,0,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> evalfr(G,0)

  130.2000

>> eig(A)

  -6.1607 +27.3022i
  -6.1607 -27.3022i
  -6.0886 +22.3260i
  -6.0886 -22.3260i
  -0.4107
  -0.0907

>> zpk(G)
```

```
             5 (s^2 + 12.5s + 525) (s^2 + 12.5s + 775)
--------------------------------------------------------------------

(s+0.4107) (s+0.09069) (s^2 + 12.18s + 535.5) (s^2 + 12.32s + 783.4)
```

```
>> y = step(G,t);
>> plot(t,y);
>>
```



Motors

4) Find the dynamics for a AM80-300 DC Servo Motor. It's specs are

- V(input) = 48VDC
- Ke = 14.1 V/krpm
- Rt = 0.67 Ohs
- Io = 0.50 Amps no load current @ 3404 rpm

When driving 1.01Nm torque

- Ia = 8.00 Amps
- w = 3048 rpm
- Pout = 322 Watts

Size = 50.00mm diameter, 89.21mm long, .

We need five parameters for

$$\omega = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)V$$

R = 0.67 Ohms

L = 0 (no information on this)

Kt:

$$\left(14.1\frac{V\cdot\min}{krev}\right)\left(\frac{1krev}{1000rev}\right)\left(\frac{60s}{1\min}\right)\left(\frac{1rev}{2\pi rad}\right) = 0.1346\frac{V}{rad/\sec}$$

D: No-load current = 0.50 Amps

3404 rpm = 356.466 rad/sec

$$P = VI = T\omega$$

$$(48V)(0.5A) = (T)\left(356.466\frac{rad}{\sec}\right)$$

$$T = 0.0673Nm = D\cdot\omega$$

$$D = \left(\frac{0.0673}{356.466}\right) = 188.9\cdot 10^{-6}\frac{Nm}{rad/\sec}$$

J: Inertia

Assume solid iron rotor, 30mm dia, 89.21mm long. The volume is

$$V = \pi r^2\cdot L = 63.06cc$$

Iron has a density of 7.87 g/cc

$$m = 496.3g$$

The rotational inertia of a solid cyllinder is

$$J = \frac{1}{2}mr^2$$

$$J = \frac{1}{2}(0.4963kg)(0.015m)^2$$

$$J = 55.83 \cdot 10^{-6} kg \cdot m^2$$

So, with L=0

$$\omega = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)V$$

$$\omega = \left(\frac{K_t}{(Js+D)(R)+K_t^2}\right)V$$

```
>> J = 55.83e-6;
>> D = 188.9e-6;
>> Kt = 0.1346;
>> R = 0.67;
>> L = 0;
>> G = tf(Kt,[J*R, D*R+Kt^2]);
>> zpk(G)

3598.3436
---------
(s+487.7)

>>
```

5) Assume this motor is used to power an electric bicycle at 20mph

- Motor speed @ 20mph = 3048 rpm @ 322 Watts
- Gear (wheel) used to convert 3048 rpm to 20mph
- Bicycle weight = 100kg

What is the gear reduction (wheel diameter) to convert 3048 rpm to 20 mph?

$$20\left(\frac{miles}{hour}\right)\left(\frac{1609m}{mile}\right)\left(\frac{1hour}{3600s}\right) = 8.941\tfrac{m}{s}$$

$$3048\left(\frac{rev}{\min}\right)\left(\frac{2\pi rad}{rev}\right)\left(\frac{1\min}{60\sec}\right) = 319.2\tfrac{rad}{\sec}$$

From

$$v = r\omega$$

$$8.941\tfrac{m}{s} = r \cdot 319.2\tfrac{rad}{s}$$

$$r = \tfrac{8.941}{319.2} = 28.01mm$$

What is the inertia relative to the DC servo motor (bring the 100kg mass back to the motor through a gear)

$$J_{bike} = r^2 \cdot m$$

$$J_{bike} = (0.02801m)^2 \cdot 100kg$$

$$J_{bike} = 0.07846$$

What is the transfer function (dynamics) for the bicycle / servo motor combination?

```
>> J = J + 0.07846;
>> G = tf(Kt,[J*R, D*R+Kt^2]);
>> zpk(G)

  2.5587
----------
(s+0.3468)
```

It now takes about 11.53 seconds to get up to speed (a reasonable number for an electric bike)

Note: The motor's inertia doesn't really matter in this calculation. Might be why the data sheets didn't provide that information...

Also also, 43V is required to go 20mph (319.2 rad/sec) is

```
>> DC = evalfr(G,0)

DC =    7.3779

>> V = 319.2 / DC

V =   43.2645
```