Homework #10: ECE 461/661

z-Transforms, s to z conversion, Root Locus in the z-Domain.   Due Monday, November 6th

z-Transforms

1) Determine the difference equation that relates X and Y

$$Y = \left(\frac{0.05z(z-1)}{(z-0.96)(z-0.92)(z-0.8)}\right)X$$

Multiply out the polynomials

```
>> P = poly([0.96,0.92,0.8])

P =    1.0000   -2.6800    2.3872   -0.7066
```

$$Y = \left(\frac{0.05z^2-0.05z}{z^3-2.68z^2+2.3872z-0.7066}\right)X$$

Cross multiply

$$(z^3 - 2.68z^2 + 2.3872z - 0.7066)Y = (0.05z^2 - 0.05z)X$$

Translate to a difference equation

y(k+3) - 2.68y(k+2) + 2.3872y(k+1) - 0.7066y(k) = 0.05x(k+2) - 0.05x(k+1)

2) Determine y(k) assuming

$$Y = \left(\frac{0.05z(z-1)}{(z-0.96)(z-0.92)(z-0.8)}\right)X \qquad x(t) = 2\cos(4t) + 3\sin(4t)$$

$$T = 0.01$$

Use phasors

$$s = j4$$

$$z = e^{sT} = 1\angle 2.291^0$$

$$X = 2 - j3$$

$$Y = \left(\frac{0.05z(z-1)}{(z-0.96)(z-0.92)(z-0.8)}\right)_{z=1\angle 2.291^0} \cdot (2 - j3)$$

$$Y = 4.9094 - j5.1887$$

meaning

$$\boxed{y(t) = 4.9094\cos(4t) + 5.1887\sin(4t)}$$

In Matlab:

```
>> G = zpk([0,1],[0.96,0.92,0.8],0.05,0.01)

        0.05 z (z-1)
-------------------------
(z-0.96) (z-0.92) (z-0.8)

Sampling time (seconds): 0.01

>> s = j*4;
>> T = 0.01;
>> z = exp(s*T);
>> X = 2-j*3;
>> Y = evalfr(G,z) * X

   Y =    4.9094 - 5.1887i
```

3) Determine y(k) assuming

$$Y = \left(\frac{0.05z(z-1)}{(z-0.96)(z-0.92)(z-0.8)}\right)X \qquad x(k) = u(k)$$

Substitute for X

$$Y = \left(\frac{0.05z(z-1)}{(z-0.96)(z-0.92)(z-0.8)}\right)\left(\frac{z}{z-1}\right)$$

Pull out a z and do partial fractions expansion

$$Y = \left(\left(\frac{0}{z-1}\right) + \left(\frac{7.50}{z-0.96}\right) + \left(\frac{-9.583}{z-0.92}\right) + \left(\frac{2.083}{z-0.8}\right)\right)z$$

Multiply through by z

$$Y = \left(\frac{0z}{z-1}\right) + \left(\frac{7.50z}{z-0.96}\right) + \left(\frac{-9.583z}{z-0.92}\right) + \left(\frac{2.083z}{z-0.8}\right)$$

Take the inverse z-transform

$$y(k) = \left(7.50(0.96)^k - 9.583(0.92)^k + 0.2083(0.8)^k\right)u(k)$$

In Matlab

```
>> Y = zpk([0,1],[1,0.96,0.92,0.8],0.05,0.01)

Zero/pole/gain:
         0.05 z (z-1)
-------------------------------
(z-1) (z-0.96) (z-0.92) (z-0.8)

Sampling time (seconds): 0.01
>> z = 1 + 1e-9;
>> evalfr(Y,z) * (z-1)

ans =  7.8125e-008

>> z = 0.96 + 1e-9;
>> evalfr(Y,z) * (z-0.96)

ans =    7.5000

>> z = 0.92 + 1e-9;
>> evalfr(Y,z) * (z-0.92)

ans =   -9.5833

>> z = 0.8 + 1e-9;
>> evalfr(Y,z) * (z-0.8)

ans =    2.0833

>>
```

s to z conversion

4) Determine the discrete-time equivalent of G(s). Assume T = 0.1 seconds

$$G(s) = \left(\frac{9111}{(s+1.21)(s+9.02)(s+23.95)(s+44.67)} \right)$$

Using Matlab, convert the poles from the s-plane to the z-plane

```
>> Ps = [-1.21,-9.02,-23.95,-44.67]'

   -1.2100
   -9.0200
  -23.9500
  -44.6700

>> T = 0.1;
>> Pz = exp(Ps*T)

    0.8860
    0.4058
    0.0912
    0.0115
```

Input G(s) and G(z). Find k to match the DC gain

```
>> Gs = zpk([],Ps,9111)

                  9111
-------------------------------------
(s+1.21) (s+9.02) (s+23.95) (s+44.67)

>> Gz = zpk([],Pz,1,0.1);
>> DC = evalfr(Gs,0)

DC =    0.7803

>> k = evalfr(Gs,0) / evalfr(Gz, 1)

k =    0.0475

>> Gz = zpk([],Pz,k,0.1)

                    0.047474
--------------------------------------------
(z-0.886) (z-0.4058) (z-0.09117) (z-0.01148)

Sampling time (seconds): 0.1
```

Now, add zeros at z=0 to match the phase (i.e. delay) at some frequency such as 1 rad/sec

```
>> s = j*1;
>> z = exp(s*T);
>> angle(evalfr(Gs,s)) * 180/pi

ans =  -49.5715
```

Target phase shift at 1 rad/sec = -49.57 degrees

With no zeros, G(z) is -64 degrees at 1 rad/sec

- target = -49.57 degrees

```
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -64.2096
```

With one zero, G(z) is -58.48 degrees

- target = -49.57 degrees

```
>> Gz = zpk([0],Pz,k,0.1);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -58.4800
```

With two zeros, G(z) is -52.75 degrees

- target = -49.57 degrees

```
>> Gz = zpk([0,0],Pz,k,0.1);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -52.7505
```

With three zeros, G(z) is -47.02 degrees

- target = -49.57 degrees

```
>> Gz = zpk([0,0,0],Pz,k,0.1);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -47.0209
```

Settle on two zeros

```
>> Gz = zpk([0,0],Pz,k,0.1)

                 0.047474 z^2
--------------------------------------------
(z-0.886) (z-0.4058) (z-0.09117) (z-0.01148)

Sampling time (seconds): 0.1
```

5) Determine the discrete-time equivalent of G(s). Assume T = 0.01 seconds

First, find the poles in the z-plane

```
>> T = 0.01;
>> Pz = exp(Ps*T)

    0.9880     0.9137      0.7870      0.6397
```

Input G(z) using these poles and find k to match the DC gain

```
>> Gz = zpk([],Pz,k,0.01)

               6.2107e-005
-----------------------------------------
(z-0.988) (z-0.9137) (z-0.787) (z-0.6397)
```

Add zeros at z=0 to match the phase (delay) at 1 rad/sec

```
>> s = j*1;
>> angle(evalfr(Gs,s)) * 180/pi

ans =  -49.5715
```

With no zeros, the phase is -50.75 degrees

```
>> T = 0.01;
>> z = exp(s*T);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -50.7550
```

With one zero, the phase is -50.18 degrees

```
>> Gz = zpk([0],Pz,k,0.01);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -50.1820
```

With two zeros, the phase is -49.60 degrees

```
>> Gz = zpk([0,0],Pz,k,0.01);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -49.6091
```

With three zeros, the phase is -49.03 degrees

```
>> Gz = zpk([0,0,0],Pz,k,0.01);
>> angle(evalfr(Gz,z)) * 180/pi

ans =  -49.0361
```

Go with two zeros (closed I could get)

```
>> Gz = zpk([0,0],Pz,k,0.01)

             6.2107e-005 z^2
-----------------------------------------
(z-0.988) (z-0.9137) (z-0.787) (z-0.6397)

Sampling time (seconds): 0.01
```

Root Locus in the z-Domain

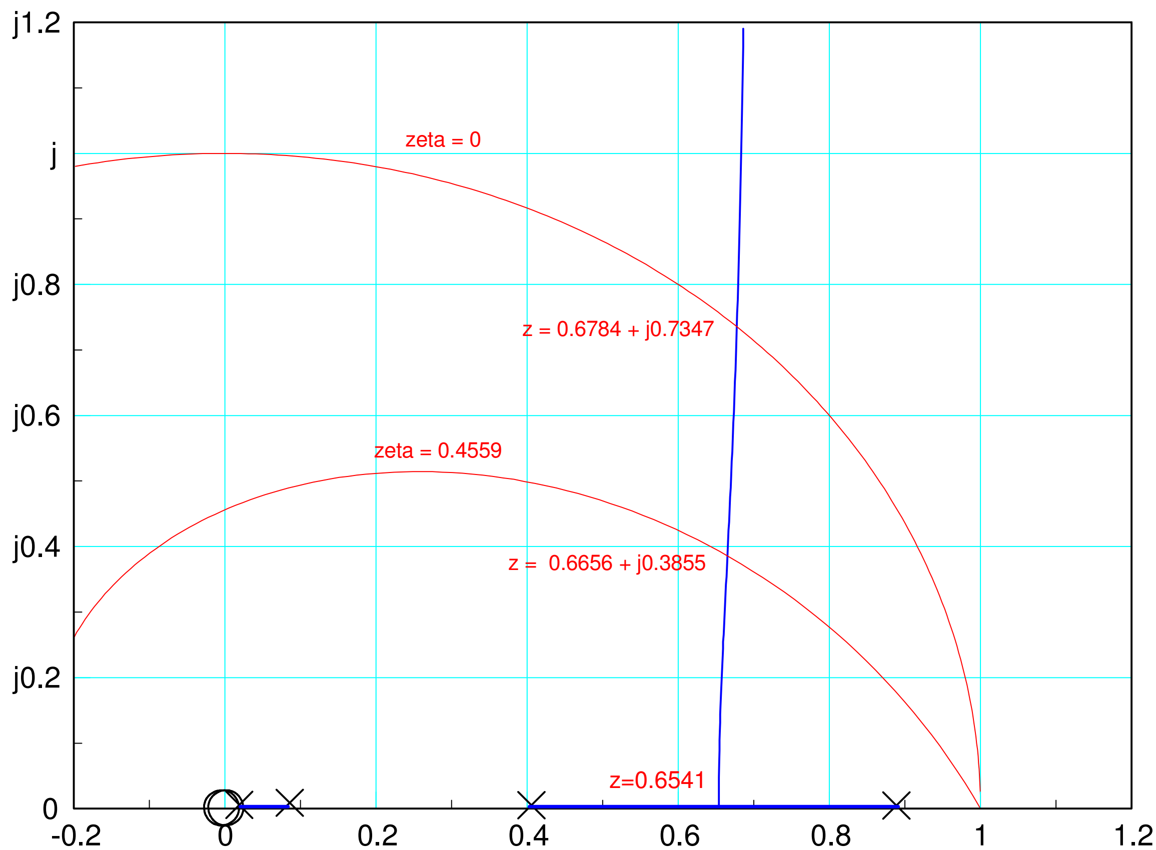
Assume T = 0.1 seconds.

$$G(s) = \left(\frac{9111}{(s+1.21)(s+9.02)(s+23.95)(s+44.67)}\right)$$

6) Draw the root locus for G(z)

From before

```
                    0.047474 z^2
-------------------------------------------------
(z-0.886) (z-0.4058) (z-0.09117) (z-0.01148)
```



7) Find k for no overshoot in the step response

- Simulate the closed-loop system's step response

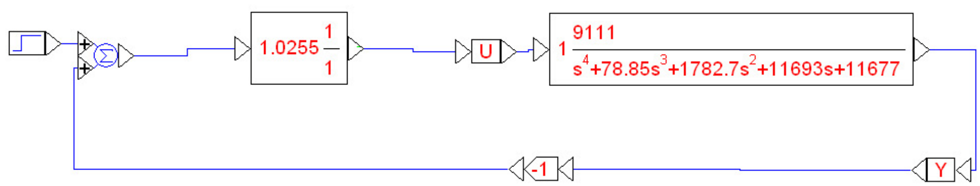
Pick a spot on the root locus (the breakaway point)

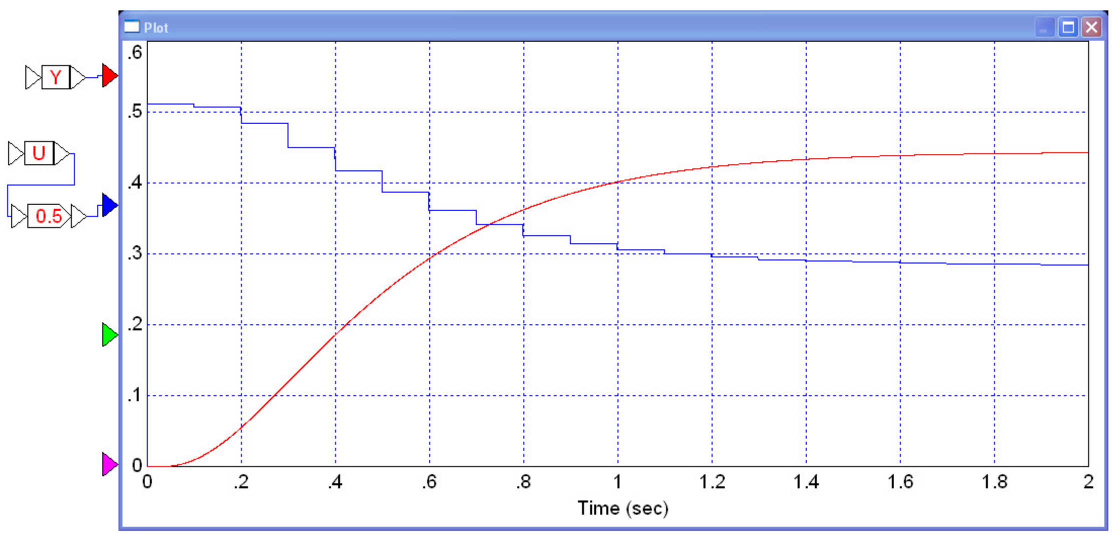
z = 0.6541

Find k so that GK = -1

```
    >> z = 0.6541;
>> evalfr(G,z)

ans =   -0.9751

>> k = 1/abs(ans)

k =    1.0255
```

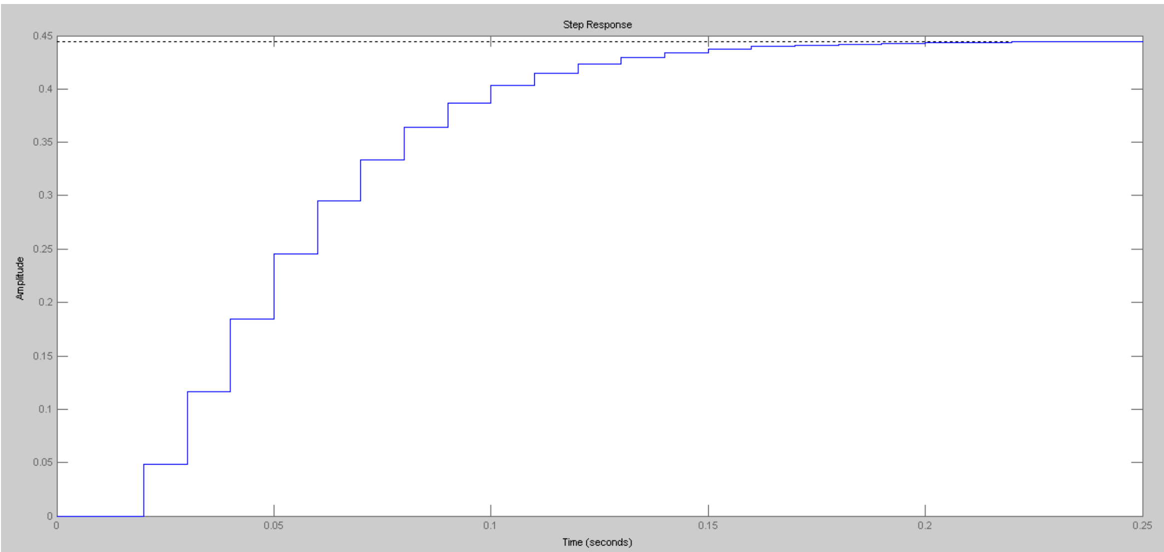
Check the closed-loop step response





In Matlab:

```
>> Gz = zpk([0,0],[0.886,0.4058,0.09117,0.01148],0.047474,0.1)

                0.047474 z^2
--------------------------------------------
(z-0.886) (z-0.4058) (z-0.09117) (z-0.01148)

Sampling time (seconds): 0.1

>> k = 1.0255;
>> Gcl = minreal(Gz*k / (1 + Gz*k))

                0.048685 z^2
---------------------------------------------
(z-0.6548) (z-0.6529) (z-0.07508) (z-0.01172)

Sampling time (seconds): 0.1

>> step(Gcl)
```



8) Find k for 20% overshoot for a step response (damping ratio = 0.4559)

- Simulate the closed-loop system's step response

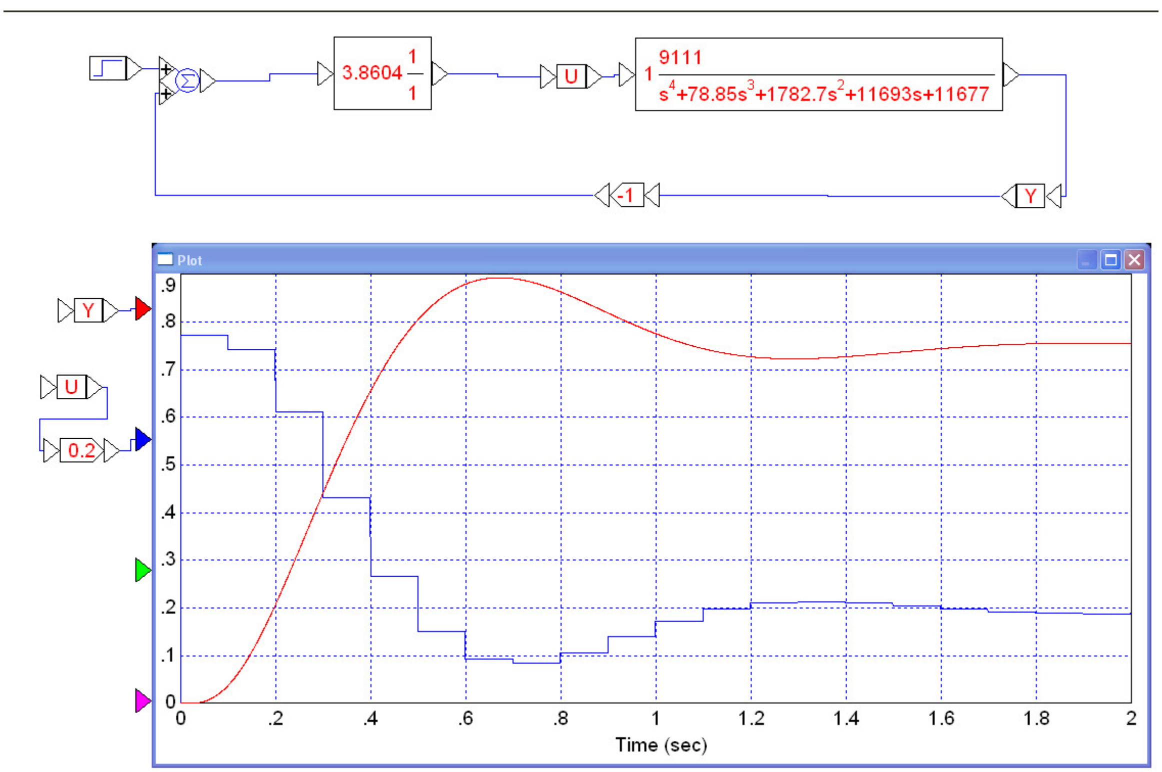
Pick a spot on the root locus

$z = 0.6656 + j*0.3855$

Find k so that GK = -1

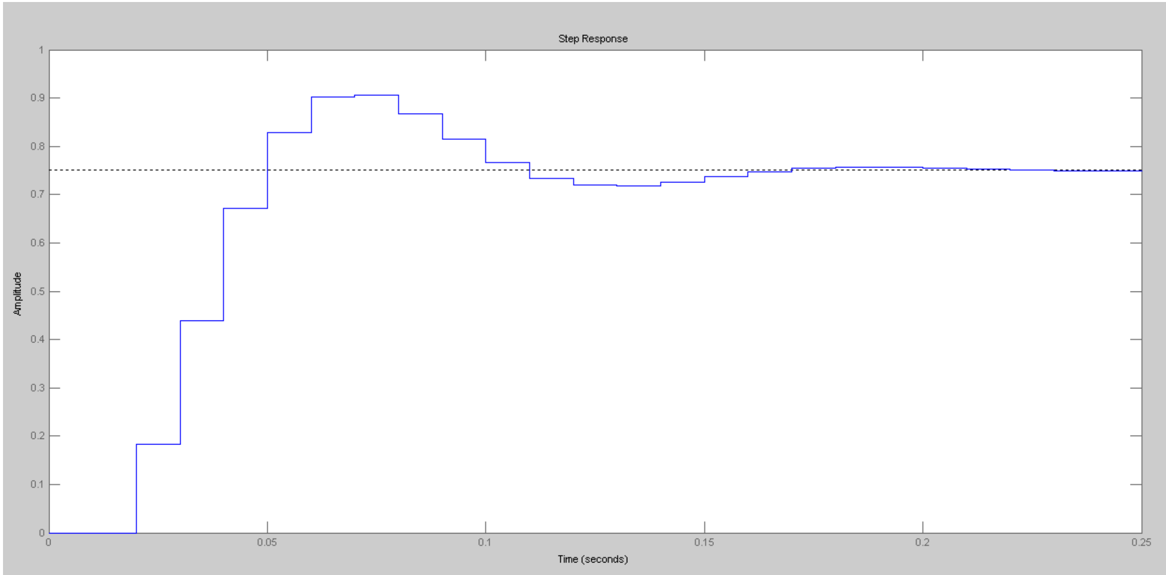
```
>> z = 0.6656 + j*0.3855;

>> evalfr(G,z)

ans =  -0.2590 + 0.0000i

>> k = 1/abs(ans)

k =    3.8604
```

Check the results:



or in Matlab

```
>> k = 3.8604;
>> Gcl = minreal(Gz*k / (1 + Gz*k))

                    0.18327 z^2
-----------------------------------------------
(z-0.05069) (z-0.01255) (z^2 - 1.331z + 0.5916)

Sampling time (seconds): 0.1

>> step(Gcl)
```


Step Response
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.05
0.1
0.15
0.2
0.25

9) Find k for a damping ratio of 0.00

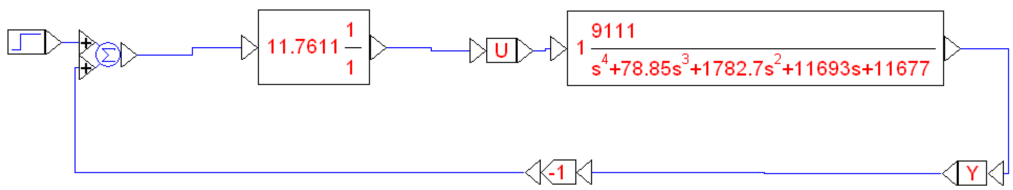
- Simulate the closed-loop system's step response

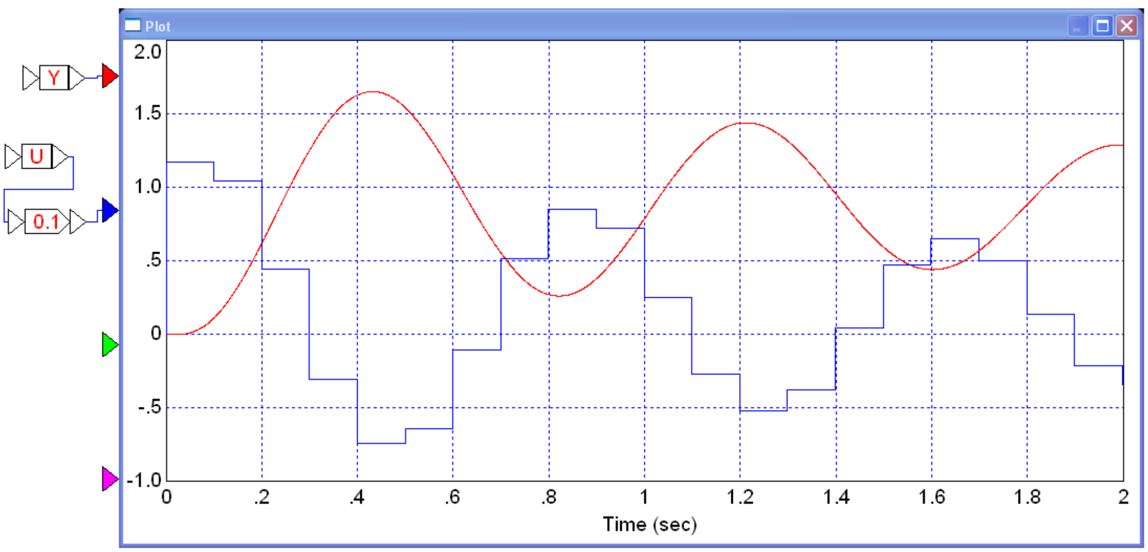
Pick a spot on the root locus

```
>> z = 0.6784 + j*0.7347;
```

Pick k to make GK = -1 at this point

```
>> evalfr(G,z)

ans =  -0.0850 - 0.0000i

>> k = 1/abs(ans)

k =   11.7611
```





Note: This is slightly stable. This is due to rounding down for the number of zeros.

- G(z) has more delay than the actual system
- By overestimating the delay, the gain chosen is slightly too low

In Matlab

```
>> k = 11.7611;
>> Gcl = minreal(Gz*k / (1 + Gz*k))

                      0.55835 z^2
----------------------------------------------------
(z^2 - 0.03775z + 0.0003763) (z^2 - 1.357z + 0.9999)

Sampling time (seconds): 0.1
```

```
>> step(Gcl)
>> xlim([0,0.2])
>>
```