First and Second Order Approximations

A transfer function is a mathemetical model which describes how a system will behave. Ideally, this model should be

- Simple, so you can understand and work with this model, and
- Accurate, so the behaviour the model predicts closely resembles how the actual system behaves.

Often times, the model for a system can be quite complex. For example, when we get to modeling systems described by the heat equation (metal bar, RC filter), the transfer function for a 10th-order RC filter with RC=1 is

$$Y = \left(\frac{1}{s^{10}+19s^9+153s^8+680s^7+1820s^6+3003s^5+3003s^4+1716s^3+495s^2+55s+1} \right) X$$

This 10th-order differential equation is a bit unwieldy. Can you come up with a simpler transfer function which behaves almost the same?

In addition, if you want to know how this system behaves, solving a 10th-order differential equations is difficult. Can you determine how this system will behave by inspection?

Fortunately, the answer is yes. Most systems have a few poles which dominate the step response. These are called *the dominant poles*. If you simplify the model of a system to

- Include the dominant poles, and
- Match the DC gain

the two systems will have approximately the same step response. Similarly, if you only consider the dominant pole, you can approximate how the system will behave by inspection.

Definitions:

Dominant Pole(s): The pole which dominates the step response of a system. These are the poles which are closest to the jw axis.

G(s): The transfer function of a system

DC Gain:
G(s) at s=0. The gain of a system for a constant input.

Settling Time: The time it takes the transients to decay to 2% of their initial value

Overshoot:
The maximum of a step response divided by it's steady-state value.

Step Response: The output of a system when a unit step is applied to the input (U(s) = 1/s)

Resonance:
The maximum gain vs. frequency, normalized by the DC gain.

Damping Ratio: The cosine of the angle of a complex pole as measured from the negative real axis.

Dominant Pole:

To determine the dominant pole of a system, find the step response of the following system:

$$G(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)} \right)$$

or, in other words, find y(t) given:

$$Y(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)} \right) \left(\frac{1}{s} \right)$$

Using partial fraction expansion

$$Y(s) = \left(\frac{2}{s} \right) + \left(\frac{-2.222}{s+1} \right) + \left(\frac{0.2469}{s+10} \right) + \left(\frac{0.0022}{s+100} \right)$$

The step response is then

$$y(t) = (2 - 2.222e^{-t} + 0.2469e^{-10t} - 0.0022e^{-100t})u(t)$$

Note that

- The first term (2) is the forced response. It remains as long as the input remains equal to 1.
- The second term $(-2.222e^{-t})$ has an initial condition 9x larger than any other term and it decays slower than the other terms.

Hence, the pole at s=-1 dominates the response (and is termed the dominant pole).

This is true in general:

The dominant pole of a system is the poles closest to s = 0

The step input $\left(\frac{1}{s} \right)$ excites poles close to s = 0. Poles close to s = 0 will likewise have the largest initial condition. Poles close to s = 0 also take longer to decay. These two together make the poles close to s = 0 dominant.

(note: There are exceptions. If a pole is unstable, it goes to infinity. All unstable poles are dominant poles. Unstable systems tend to break, however, so we don't normally deal with them.)

Dominant poles also let you simplify the system For example, determine a 1st-order approximation for G(s).

To do this

- Keep the dominant pole (s = -1), and
- Match the DC gain

The dominant pole was s = -1, so

$$G(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)} \right) \approx \left(\frac{a}{s+1} \right)$$

The constant 'a' is selected so that G(s=0) is the same in both cases. Setting the DC gain to be the same

$$G(s) \approx \left(\frac{2}{s+1} \right)$$

You can check the approximation in Matlab. First, input the system. You can do this in zeros / poles / gain form:

```
>> G3 = zpk([],[-1,-10,-100],2000)

        2000
-------------------
(s+1) (s+10) (s+100)
```

or in transfer function form:

```
>> poly([-1,-10,-100])

          1        111       1110       1000

>> G3 = tf(2000,[1,111,1110,1000])

          2000
---------------------------
s^3 + 111 s^2 + 1110 s + 1000
```

Once you have G3, you can convert from one form to the other:

```
>> zpk(G3)

        2000
-------------------
(s+100) (s+10) (s+1)


>> tf(G3)

          2000
---------------------------
s^3 + 111 s^2 + 1110 s + 1000
```

A first order approximation has a pole at s = -1 (the dominant pole) and matches the DC gain
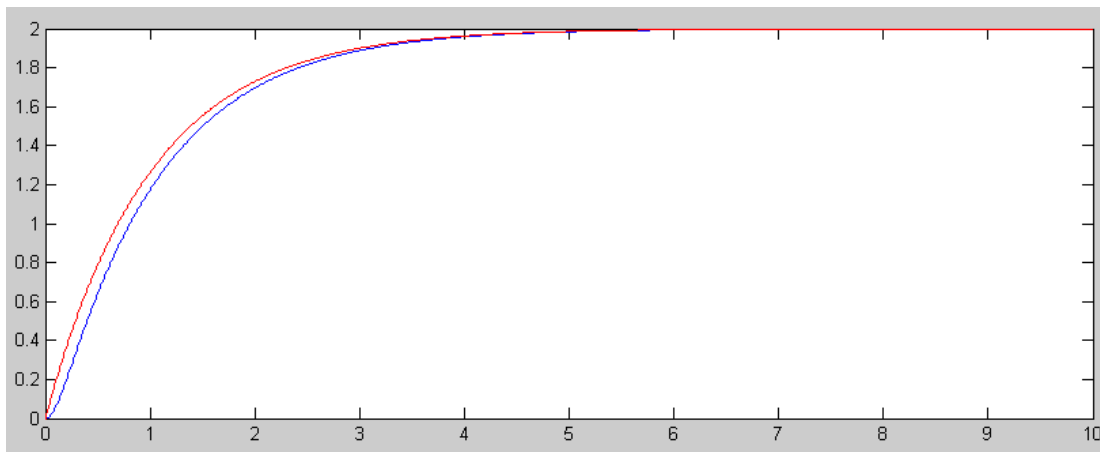
```
>> DC = evalfr(G3,0)

DC =    2
```

so, a 1st-order approximation is:

```
>> G1 = zpk([],-1,2)

   2
-----
(s+1)
```

Checking the step response in Matlab:

```
>> t = [0:0.01:10]';
>> y1 = step(G1,t);
>> y3 = step(G3,t);
>> plot(t,y3,'b',t,y1,'r')
```

Step Response of G(s) (blue) and its 1st-Order Approximation (red)

This also works with complex poles. For example, find a simplified model for a system with complex poles:

$$G(s) = \left(\frac{2,000,000}{(s+1+j2)(s+1-j2)(s+10)(s+50+j200)(s+50-j200)} \right)$$

Solution: The dominant pole is the pole which decays the slowest and is closest to s=0 are (-1+j2, -1-j2). Since the pole is complex, its complex conjugate will also be present in any model. Both decay at the same rate, so there are two dominant poles.

The simplified model is then

$$G(s) = \left(\frac{2000}{(s+1+j2)(s+1-j2)(s+10)(s+50+j200)(s+50-j200)} \right) \approx \left(\frac{a}{(s+1+j2)(s+1-j2)} \right)$$

Setting the DC gain to be the same, 'a' is found to be 0.000941

$$G(s) \approx \left(\frac{0.000941}{(s+1+j2)(s+1-j2)} \right)$$

Checking the step responses in Matlab:

Input the system into Matlab:

```
>> G5 = zpk([],[-1+j*2,-1-j*2,-10,-50+j*200,-50-j*200],2e6)

                    2000000
-------------------------------------------
(s+10) (s^2 + 2s + 5) (s^2 + 100s + 4.25e004)

>> DC = evalfr(G5,0)

    0.941176470588236
```

Since the dominant pole is complex, there are two dominant poles. Form a system with the same dominant poles. For now, guess '1' for the numerator. This will result in the DC gain being off, but we'll adjust that later
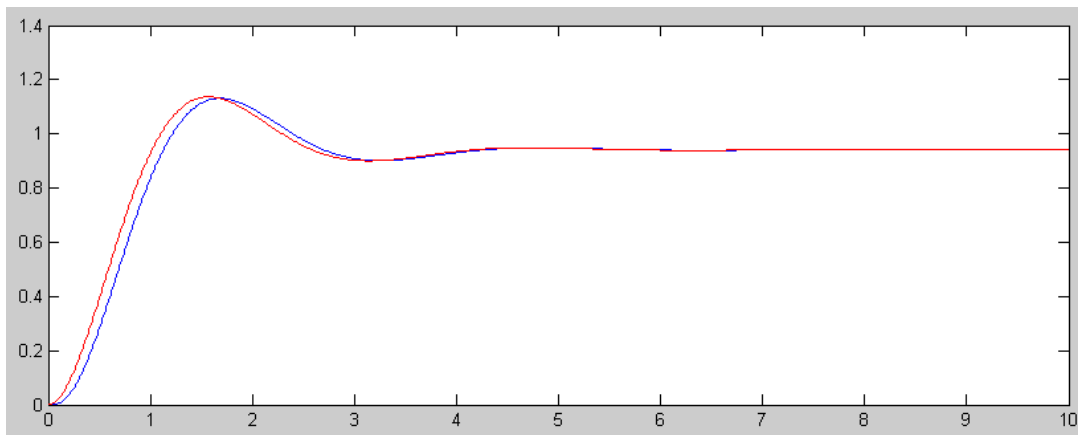
```
>> G2 = zpk([],[-1+j*2,-1-j*2],1)

       1
--------------
(s^2 + 2s + 5)

>> evalfr(G2,0)

   0.200000000000000
```

To make the DC gains match, change the numerator to 4.7059

```
>> a = DC / evalfr(G2,0)

   4.705882352941178

>> G2 = zpk([],[-1+j*2,-1-j*2],a)

     4.7059
--------------
(s^2 + 2s + 5)
```

 The two step responses are then
```
>> y5 = step(G,t);
>> y2 = step(G2,t);
>> plot(t,y5,'b',t,y2,'r')
>>
```


Step Response of the 5th-Order System and its 2nd-Order Approximation

Time Scaling

Since a transfer function is only a model that simulates a real system, it is not necessary for one second in simulation time to correspond to one second in real time. Sometimes, it is an advantage to have different time scales. For example, if modeling the effect of interest rates on the U.S. economy, it would be very convenient if 1 month real time corresponded to 1 second in simulation time. Events which take effect in 6 months could then be observed in 6 seconds. Fast events can also be slowed down so that you can observe what is going on (such as modeling the propagation of a flame or pressure wave).

In order to time scale a transfer function

i) Define the relationship between simulation time (τ) and real time (t) as $\tau = \alpha t$.

- For example, if 1s simulation time corresponds to 1ms real time (i.e. the simulation is 1000x slower than the actual system), $\tau = 0.001t$.

ii) Substitute for $s \rightarrow \alpha s$

- The LaPlace transform assumes all functions are in the form of e^{st}. Substituting for t results in all functions being in the form of $e^{(s/\alpha)\tau}$.
- To slow up the system by 1000, replace s with $\frac{s}{0.001} = 1000s$.

iii) Simplify the transfer function.

Example: The dynamics of a servo-motor are given as $G(s) = \left(\frac{10^7}{s(s+500)(s+2000)} \right)$. Find the transfer function if time scaled by 1000x (i.e. the model is 1000x slower than the actual system.)

Solution: Replace 's' with $1000s$

$$G(s) \Rightarrow \left(\frac{10^7}{(1000s)(1000s+500)(1000s+2000)} \right) = \left(\frac{0.01}{s(s+0.5)(s+2)} \right)$$

note: In this course, most systems will have poles with a magnitude close 1. This is done since i) with time scaling, any system can be modeled as one with poles close to 1, and ii) 1 has nice numerical properties. (i.e. $1^n=1$)

First-Order Approximations:

Assume the simplified model for a system is a first-order system. Given this model, try to predict how the system will behave.

This is pretty simple. A generic 1st-order system is

$$G(s) = \left(\frac{a}{s+b} \right)$$

DC Gain: The DC gain is the gain at s = 0:

$$DC = \left(\frac{a}{b} \right)$$

2% Settling Time: If you take the step response

$$Y = \left(\frac{a}{s+b} \right)\left(\frac{1}{s} \right)$$

you get

$$y(t) = \left(\frac{a}{b} \right)(1 - e^{-bt})u(t)$$

If you ask how long does this take to get to its final value (1.000), the answer is infinity. Saying the settling time is infinte doesn't help much. Instead, the standard is to ask

When does y(t) get to and remain within 2% of its final value?

or, when is

$$e^{-bt} = 0.02$$

Solving

$$t_{2\%} = \frac{3.912}{b} \approx \frac{4}{b}$$

The 2% settling time is 4 / the real part of the dominant pole

Going back to the 10th-order heat equation we started with,

$$Y = \left(\frac{1}{s^{10}+19s^9+153s^8+680s^7+1820s^6+3003s^5+3003s^4+1716s^3+495s^2+55s+1} \right) X$$

The DC gain tells you where the system winds up

$$DC = \left(\frac{1}{s^{10}+19s^9+153s^8+680s^7+1820s^6+3003s^5+3003s^4+1716s^3+495s^2+55s+1} \right)_{s=0} = 1$$

```
>> DC = evalfr(G10,0)

    1.0000
```

The dominant pole tells you the 2% settling time:

```
>> zpk(G10)

Zero/pole/gain:
                                                    1
-------------------------------------------------------------------------------------------------
(s+3.911) (s+3.652) (s+3.247) (s+2.731) (s+2.149) (s+1.555) (s+1) (s+0.5339) (s+0.1981) (s+0.02234)
```

The dominant pole is at s = -0.02234.
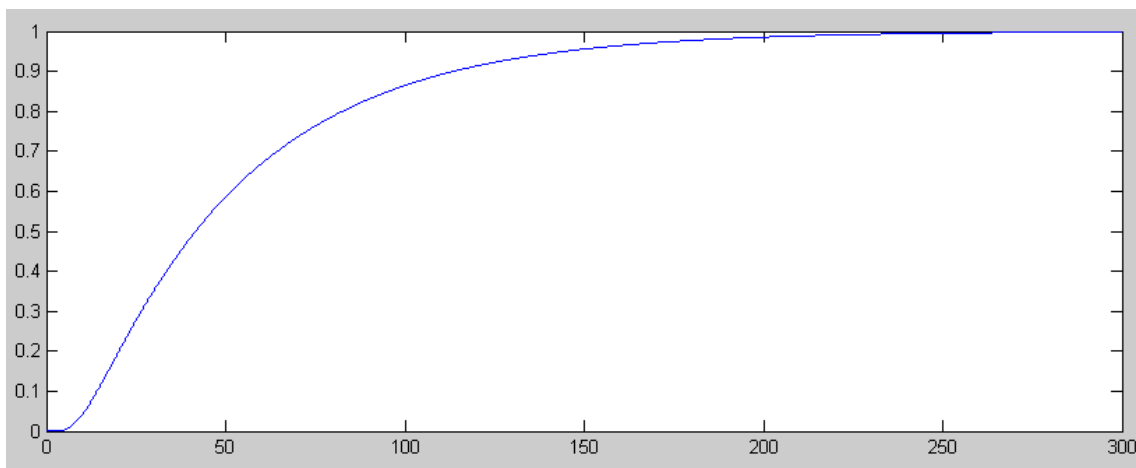
The 2% settling time is

$$t_{2\%} = \frac{4}{0.02234} = 179.05 \text{ seconds}$$

This system should have a step response
- Which goes to 1.000, and
- Which takes 179 seconds to settle out.

Checking in Matlab:

```
>> t = [0:0.1:300]';
>> y10 = step(G10,t);
>> plot(t,y10);
```


As expected, the 10th-Order RC Filter has a DC gain of one and a 2% settling time of about 179 seconds

This also works the other way: given the step response, you can determine G(s). For the above plot, for example,

- The DC gain is 1.00
- The 2% settling time is 178 seconds (approximately).

This tells you that the dominant pole is

$$b \approx \frac{4}{178} = 0.0225$$

$$G(s) \approx \left(\frac{a}{s+0.0225} \right)$$

To match the DC gain

$$\left(\frac{a}{s+0.0225} \right)_{s=0} = 1$$

$$a = 0.0225$$

and

$$G(s) \approx \left(\frac{0.0225}{s+0.0225} \right)$$

Note that when you determine the transfer funciton from the step response, you only get the dominant pole. That's sort of why it's called *the dominant pole*.

Second-Order Approximations:

Assume the simplified model for a system is second-order:

$$G(s) = \left(\frac{ac}{s^2 + bs + c} \right)$$

Factoring the denominator and placing it in rectangular form gives

$$G(s) = \left(\frac{a\omega_n^2}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} \right)$$

If you take the step response of this system, the terms will be of the form

$$y(t) = a + be^{-\sigma t}\cos(\omega_d t + \phi) \quad (t > 0)$$

where 'b' and ϕ are constants. Note that the step response can be determined by inspection by looking at the different terms in the transfer function:

- **The real part of the dominant pole determines the 2% settling time**
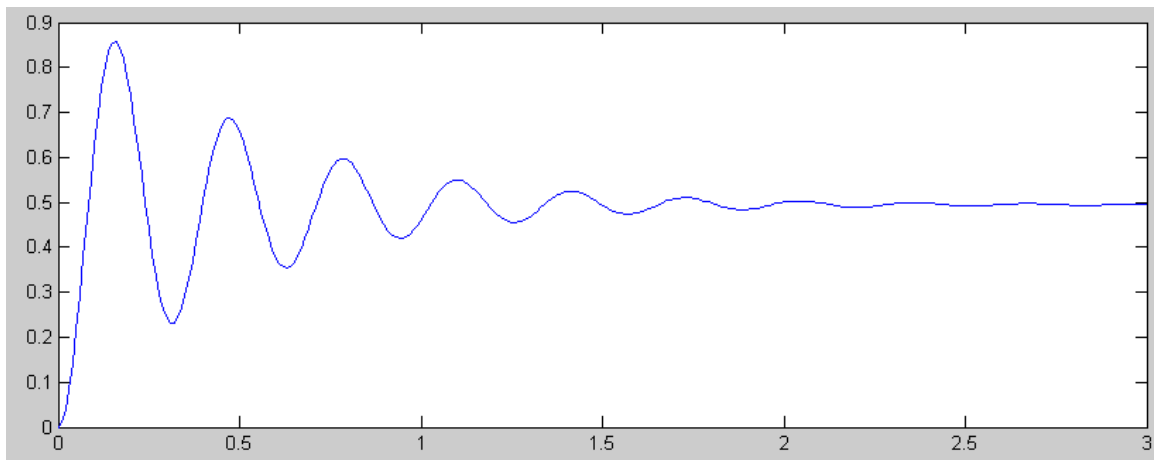- **The complex part of the pole determines the frequency of oscillation**

Example: Determine how $G(s) = \left(\frac{200}{(s + 2 + j20)(s + 2 - j20)} \right)$ will behave:

Solution:

- The step response will eventually go to 0.495. (G(0) = 0.495).
- It will take approximately 2 seconds to reach steady-state. (The real part of the dominant pole is -2. Hence, the transient decays as e^{-2t}), and
- The transient will oscillate at 20 rad/sec (the complex part of the pole is j20).

The actual step response follows:

```
>> t = [0:0.01:3]';
>> y2 = step(G2,t);
>> plot(t,y2);
```


Step Response: The 2% settling time is 2.00 seconds and the frequency of oscillation is 20 rad/sec (3.18 Hz)

The overshoot is a function of the angle of the dominant pole. If you express the pole in polar form, then

$$G(s) = \left(\frac{a\omega_n^2}{(s+\omega_n\angle\theta)(s+\omega_n\angle-\theta)} \right)$$

or

$$G(s) = \left(\frac{a\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \right)$$

where $\zeta = \cos\theta$. ζ (zeta) is called *the damping ratio*. This determines the overshoot for a step input:
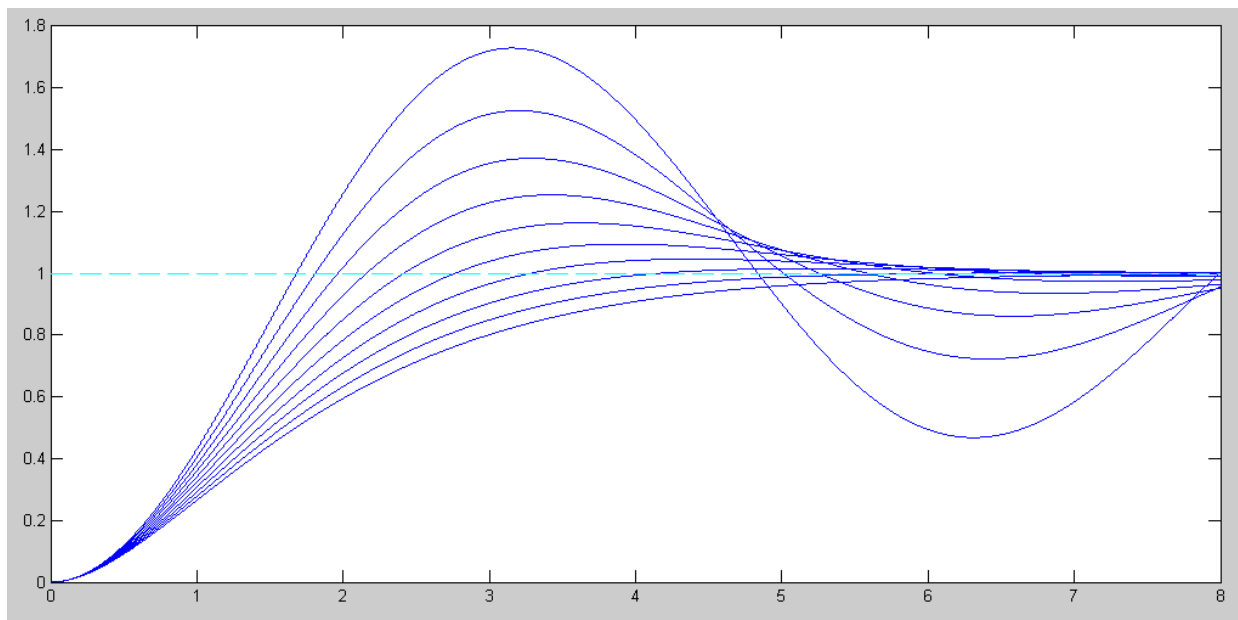
$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \right)$$

Esentially, the angle of the dominant pole determines the overshoot for a step input:

ζ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
θ	90^0	84.3^0	78.5^0	72.5^0	66.4^0	60^0	53.1^0	45.6^0	36.9^0	25.9^0	0^0
OS	100%	72.9%	52.7%	37.2%	25.4%	16.3%	9.5%	4.6%	1.5%	0.2%	0%

The effect of the angle can be seen on the following plot where the step response for different damping rations is plotted:

$$G(s) = \left(\frac{1}{(s+1\angle\theta)(s+1\angle-\theta)} \right) = \left(\frac{1}{s^2+2\zeta s+1} \right)$$

Step Response for a Damping Ratio of { 0.1 0.2 0.3   ... 1.0 }

Example: Determine the step response of the following system by inspection:

$$G(s) = \left(\frac{2000}{(s+1+j2)(s+1-j2)(s+10)(s+50+j200)(s+50-j200)} \right)$$

Solution: The DC gain is 0.94

The dominant pole is at -1 + j2

- The 2% settling time will be 4 seconds   (4/1)
- The frequency of oscillation will be 2 rad/sec
- The overshoot will be 20.79%

$$1+j2 = 2.23\angle 63.4^0$$

$$\zeta = \cos(63.4^0) = 0.447$$

$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \right) = 20.79\%$$
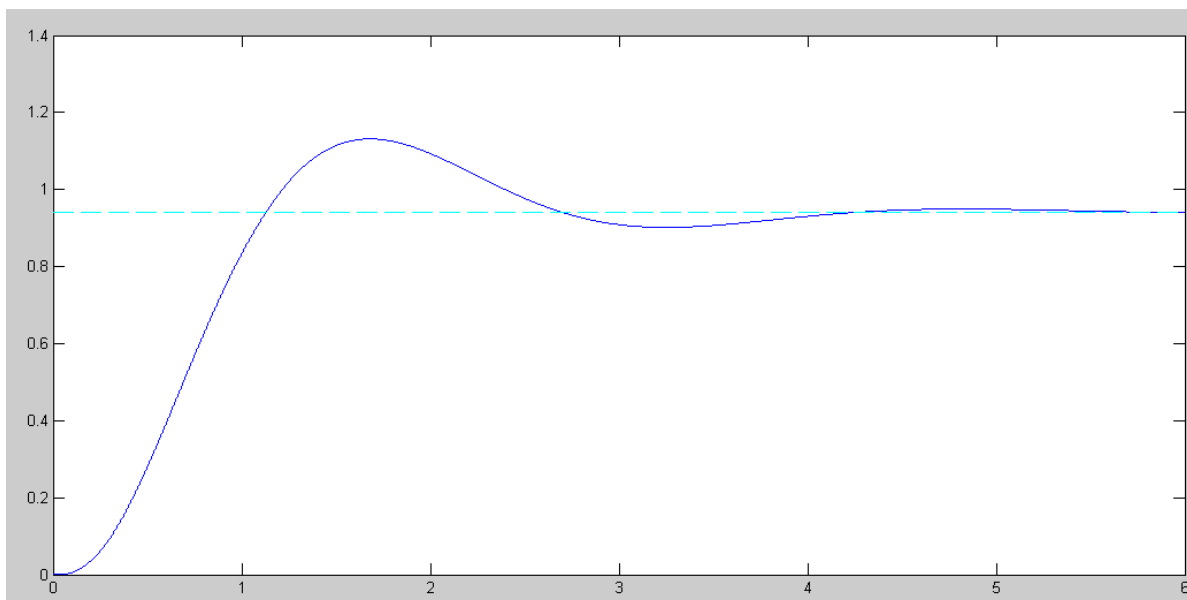
The actual step response is:

```
>> G = zpk([],[-1+j*2,-1-j*2,-10,-50+j*200,-50-j*200],2e6);
>> t = [0:0.001:6]';
>> y = step(G,t);
>> DC = evalfr(G,0)

DC =   0.9412

>> plot(t,y,'b',[0,6],[1,1]*DC,'c--')
>> OS = max(y) / DC

OS =   1.2021
```


Step Response of G(s): DC gain = 0.94, 20% Overshoot, and t(2%) = 4.0 seconds

This also works the other way: Given the above step response, determine G(s).

The DC gain is 0.9412

The 2% settling time is 4 seconds, meaning the real part of the dominant pole is -1

The overshoot is 20.21%, meaning

$$\zeta = 0.4536$$

$$\theta = \arccos{(\zeta)} = 63.02^0$$

$$s = -1 + j1.96$$

so

$$G(s) \approx \left(\frac{a}{(s+1+j1.96)(s+1-j1.96)} \right)$$

Matching the DC gain

$$\left(\frac{a}{(s+1+j1.96)(s+1-j1.96)} \right)_{s=0} = 0.94$$

$$a = 4.55$$

so

$$G(s) \approx \left(\frac{4.55}{(s+1+j1.96)(s+1-j1.96)} \right)$$

Note again that you only pick out the dominant pole(s) when you find the transfer function from the step response.