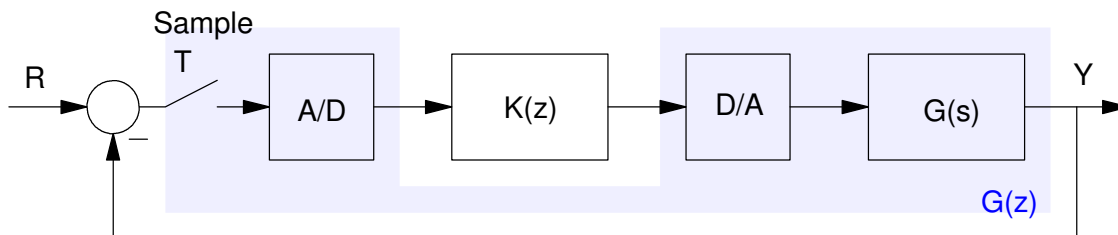


Converting G(s) to G(z)

Often times, the compensator, K(s), will be implemented with a microcontroller. Relative to the microcontroller, the world looks discrete:

- You output a control signal every T seconds,
- Through the system dynamics, this results in an error, sampled every T seconds as well.

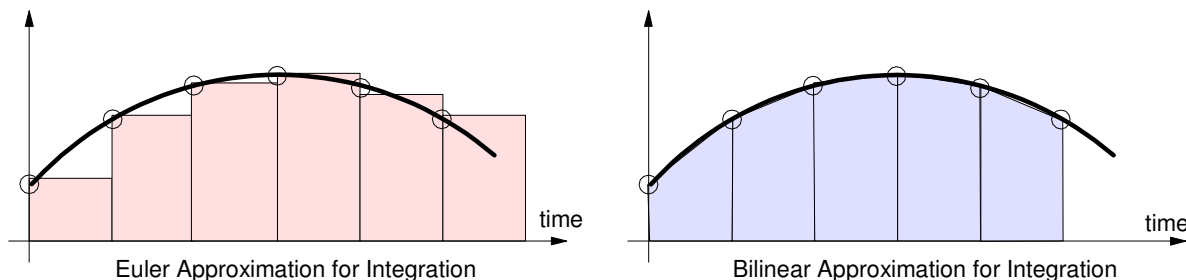


This is a hybrid system - both s-plane (for the plant) and z-plane (for the compensator) - which makes analysis difficult. Since we are designing the compensator, K(z), it makes some sense to convert the system to the z-plane.

The problem then becomes, how do you convert a system, G(s), into the z-plane, creating G(z)?

Method 1: Substitution

Substitute a discrete-time approximation for s. One way it to come up with a form of numerical integration (determining the area under the curve) then relating 1/s (integration) to your algorithm. Two common forms of numerical integration are Euler and Bilinear



These result in the following approximations for 's'

Euler - Backward Difference

$$s \approx \left(\frac{z-1}{T} \right)$$

Euler - Forward Difference:

$$s \approx \left(\frac{z-1}{Tz} \right)$$

Bilinear

$$s \approx \frac{2}{T} \left(\frac{z-1}{z+1} \right)$$

Example: Find the z-transform of G(s) assuming a sampling rate of 100ms (T = 0.1).

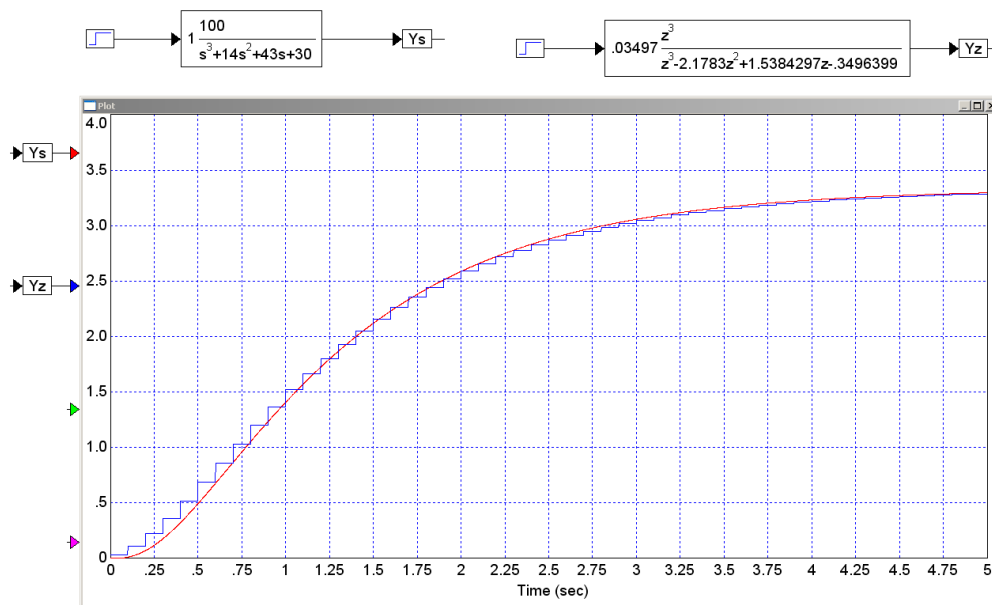
$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

a) Using Euler Forward Difference:

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

$$G(z) \approx \left(\frac{100}{\left(\left(\frac{z-1}{Tz} \right) + 1 \right) \left(\left(\frac{z-1}{Tz} \right) + 3 \right) \left(\left(\frac{z-1}{Tz} \right) + 10 \right)} \right)$$

$$G(z) = \left(\frac{0.03497z^3}{(z-0.9091)(z-0.7692)(z-0.5)} \right)$$



Step Response of G(s) and G(z) Using Euler-Forward Difference

In Matlab: Input the system in the s-plane

```
Gs = zpk([], [-1, -3, -10], 100)

      100
-----
(s+1) (s+3) (s+10)
```

If you add one more term, Matlab interprets this as a discrete-time system with the last term being the sampling rate:

```
Gz = zpk([0, 0, 0], [0.9091, 0.7692, 0.5], 0.03497, 0.1)

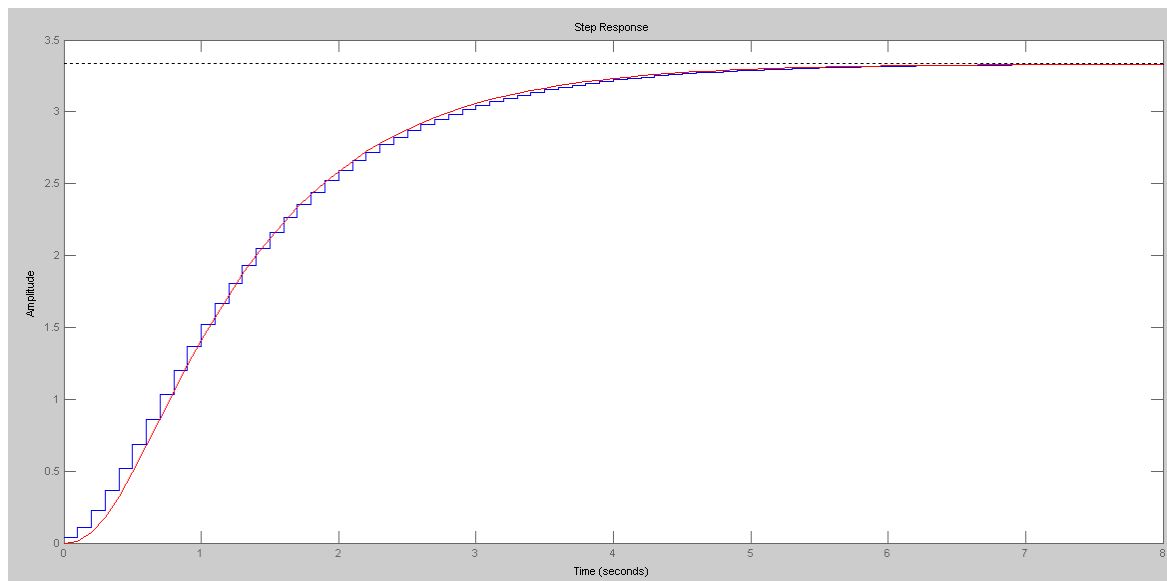
      0.03497 z^3
-----
(z-0.9091) (z-0.7692) (z-0.5)
Sampling time (seconds): 0.1
```

To plot the step response of the two systems together

- Plot the step response of $G(z)$
- Type *hold on* to keep this plot
- Plot the step response of $G(s)$ on top of the $G(z)$ graph

```
step(Gz)

t = [0:0.01:8]';
ys = step(Gs, t);
hold on
plot(t, ys, 'r')
```



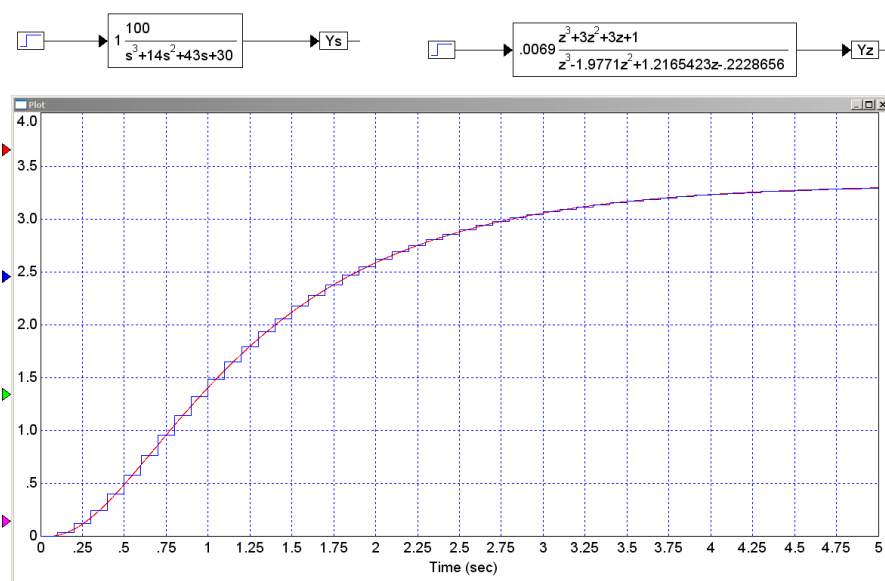
Bilinear:

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

$$G(z) \approx \left(\frac{100}{\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+1\right)\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+3\right)\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+10\right)} \right)$$

$$G(z) \approx \left(\frac{100T^3(z+1)^3}{(2(z-1)+T(z+1))(2(z-1)+3T(z+1))(2(z-1)+10T(z+1))} \right)$$

$$G(z) \approx \left(\frac{0.00690(z+1)^3}{(z-0.9047)(z-0.7391)(z-0.3333)} \right)$$



Step Response of G(s) and G(z) Using Euler-Forward Difference

Method #2: Transform Poles and Zeros (my preference)

LaPlace transforms assume everything is in the form of

$$y = e^{st}$$

z-transforms assume everything is in the form of

$$y = z^k$$

Assume a sampled data system where

$$t = kT$$

Substituting

$$y = e^{s(kT)} = e^{(sT)k} = (e^{sT})^k = z^k$$

The transformation from s-plane to z-plane (and back) is

$$z = e^{sT}$$

This leads to the following method:

i) Convert every pole and zero in the s-plane to a pole and zero in the z-plane as

$$z = e^{sT}$$

ii) Add a gain to match the gain of $G(s)$ and $G(z)$ at some frequency - typically DC

iii) (optional) Add n zeros at $z = 0$ to match the phase at a frequency close to zero

- or -

Add n zeros at $z = 0$ to match the delay in the system.

Example: Find the discrete-time equivalent of $G(s)$ with a sampling rate of $T = 0.1$

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

In Matlab

```
-->s = [-1;-3;-10]
```

```
- 1.
- 3.
- 10.
```

```
-->T = 0.1
```

```
0.1
```

```
-->z = exp(s*T)
```

```
0.9048374
0.7408182
0.3678794
```

meaning

$$G(z) = \left(\frac{k}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)$$

To find k , match the gain at DC

$$\left(\frac{100}{(s+1)(s+3)(s+10)} \right)_{s=0} = 3.3333$$

$$\left(\frac{k}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)_{z=1} = 3.33333$$

$$k = \text{prod}(1-z) * 3.33333 \\ 0.0519691$$

meaning

$$G(z) = \left(\frac{0.051969}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)$$

To find how many zeros belong at $z=0$,

- There is too much delay with this system. Add zeros until the step response is "close"
- Match the phase at some frequency, such as $s = j1$

$$\left(\frac{100}{(s+1)(s+3)(s+10)} \right)_{s=j1} = 2.2249 \angle -69.14^\circ$$

$G(z)$ at $s = j1$ is:

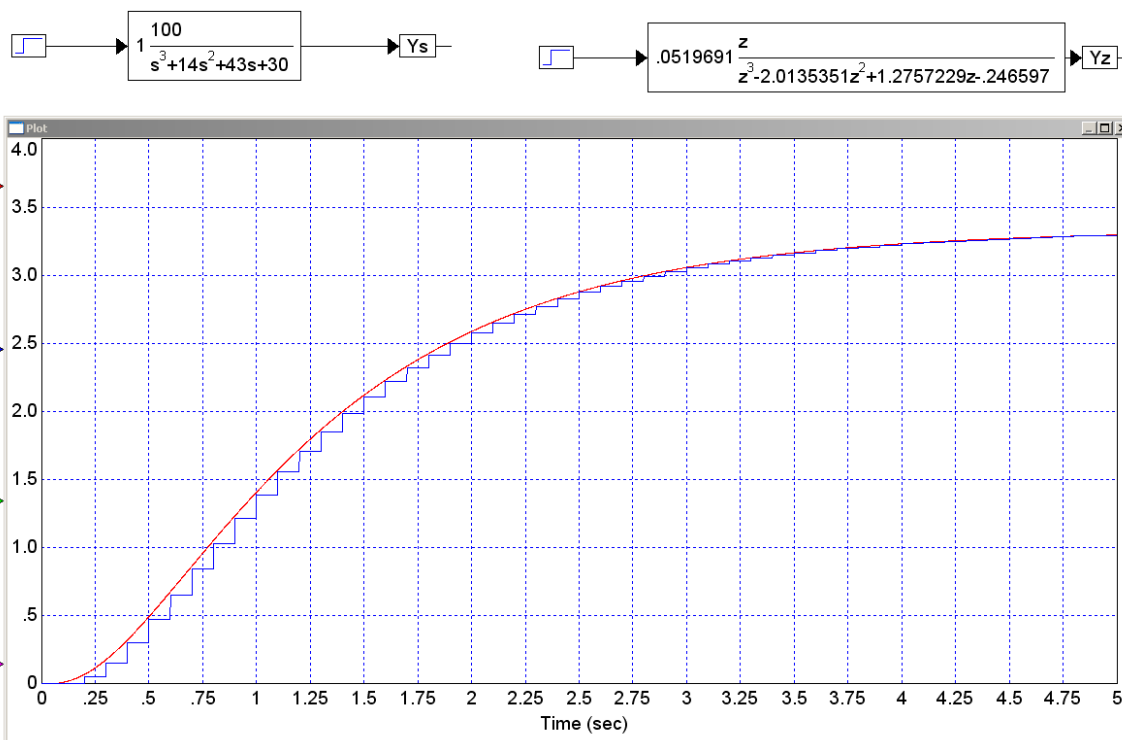
$$\left(\frac{0.051969}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)_{s=j} = 2.2276 \angle -78.04^\circ$$

so the phase is off by 8.90 degrees. Each zero at $z = 0$ adds 5.73 degrees

$$e^{sT} = e^{(j1)(0.1)} = 1 \angle 5.73^\circ$$

meaning you need to add 1.55 zeros. I don't know how to add a fraction of a zero, so add one zero at $z=0$, resulting in

$$G(z) \approx \left(\frac{0.051969 \cdot z}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)$$



In Matlab: Input the system G(s)

```
Gs = zpk([], [-1, -3, -10], 100)

      100
-----
(s+1) (s+3) (s+10)
```

Input G(z). For now, assume the numerator is 1

```
T = 0.1;
Gz = zpk([], [exp(-1*T), exp(-3*T), exp(-10*T)], 1, T)

      1
-----
(z-0.9048) (z-0.7408) (z-0.3679)

Sampling time (seconds): 0.1
```

Add a gain, k, so that the DC gain matches up

```
DCs = evalfr(Gs, 0)

3.3333
```

```
DCz = evalfr(Gz,1)
```

```
64.1401
```

```
k = DCs / DCz
```

```
0.0520
```

So, $G(z)$ is.

```
Gz = zpk([], [exp(-1*T), exp(-3*T), exp(-10*T)], k, T)
```

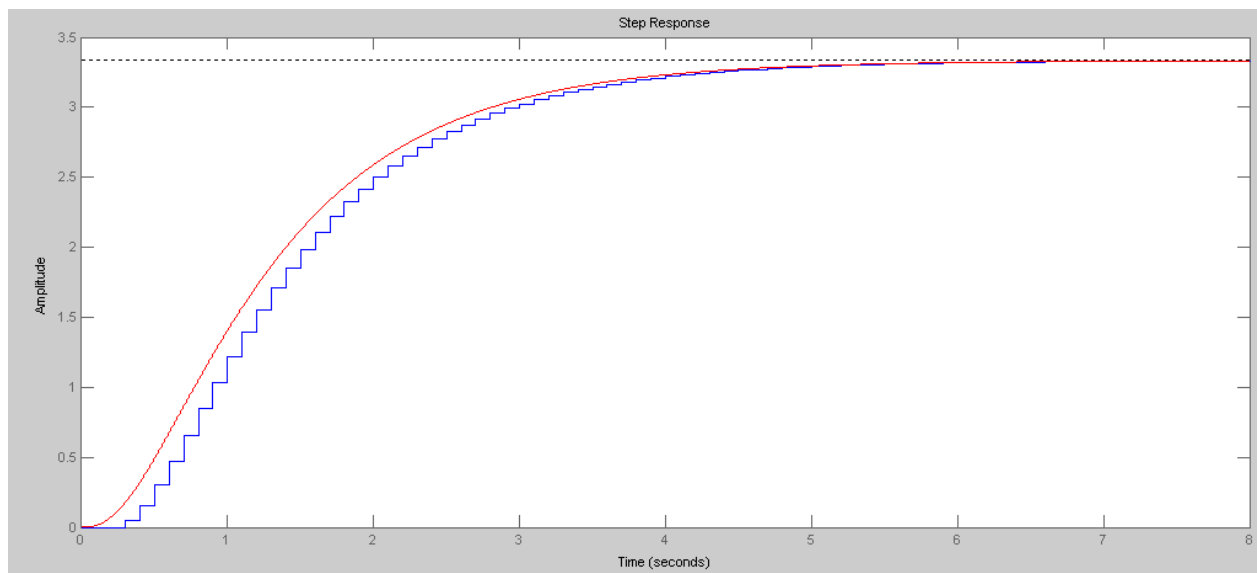
```
0.05197
```

```
-----  
(z-0.9048) (z-0.7408) (z-0.3679)
```

```
Sampling time (seconds): 0.1
```

Checking the answer: Plot the step response of $G(z)$ and $G(s)$

```
step(Gz)
hold on
t = [0:0.001:8]';
ys = step(Gs,t);
plot(t,ys,'r');
```



Note that there is too much delay in $G(z)$. Adding one or two zeros at $z=0$ will help

Example 2: Complex Poles

This also works with complex poles and zeros

Determine a discrete-time system, $G(z)$, which behaves approximately the same as $G(s)$

$$Y = \left(\frac{100(s+j5)(s-j5)}{(s+1+j4)(s+1-j4)(s+2)} \right)$$

$$sn = [j*5, -j*5]'$$

$$T = 0.1$$

$$zn = \exp(ns*T)$$

$$0.8775826 - 0.4794255i$$

$$0.8775826 + 0.4794255i$$

poly(nz)

$$1. \quad - 1.7551651 \quad 1.$$

$$sd = [-1+j*4, -1-j*4, -20]'$$

$$- 1. \quad - 4.i$$

$$- 1. \quad + 4.i$$

$$- 20.$$

$$zd = \exp(sd*T)$$

$$0.8334105 - 0.3523603i$$

$$0.8334105 + 0.3523603i$$

$$0.1353353$$

poly(zd)

$$1. \quad - 1.8021562 \quad 1.0443104 \quad - 0.1108032$$

meaning

$$G(z) = k \left(\frac{z^2 - 1.755z + 1.}{z^3 - 1.802z^2 + 1.044z - 0.110} \right)$$

To find k, match the DC gain:

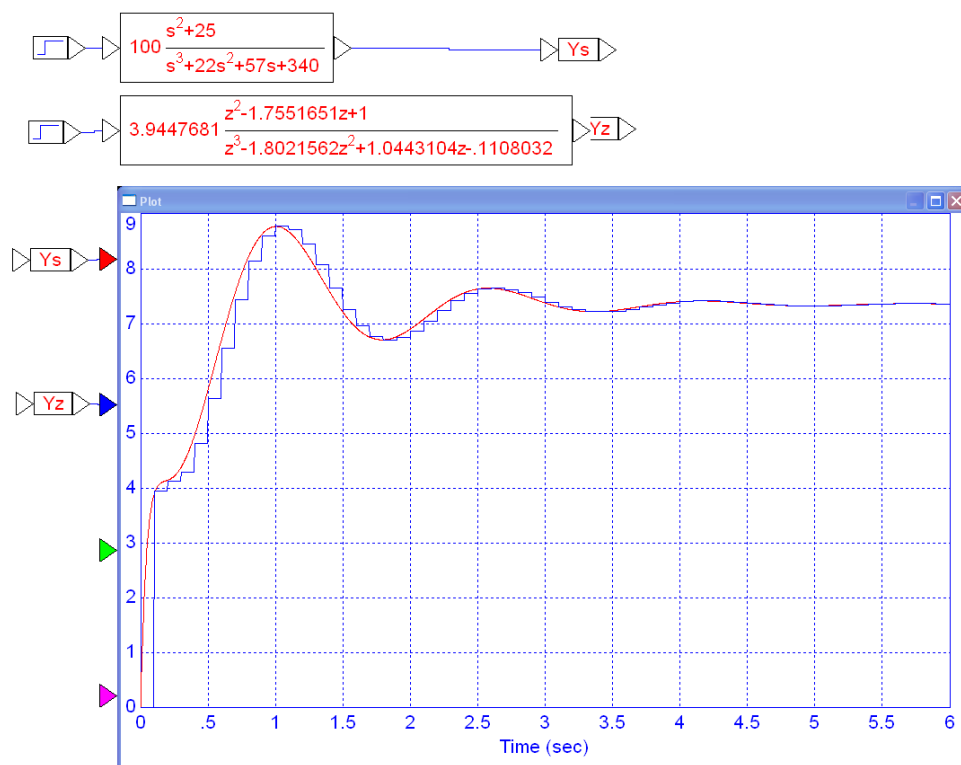
$$DC = 100 * 25 / 340$$

$$7.3529412$$

$$k = DC * \text{prod}(1-zd) / \text{prod}(1-zn)$$

$$3.9447681$$

$$G(z) = 3.944 \left(\frac{z^2 - 1.755z + 1.}{z^3 - 1.802z^2 + 1.044z - 0.110} \right)$$



Step Response of G(s) and G(z)

In Matlab: Input the system G(s)

```
z1 = j*5;
z2 = -j*5;
p1 = -1+j*4;
p2 = -1-j*4;
p3 = -2;
Gs = zpk([z1, z2], [p1, p2, p3], 100)
```

$$\frac{100 (s^2 + 25)}{(s+2) (s^2 + 2s + 17)}$$

Now input G(z). Convert the poles and zeros to the z-plane as e^{sT} .

```
T = 0.1;
Gz = zpk([exp(z1*T), exp(z2*T)], [exp(p1*T), exp(p2*T), exp(p3*T)], 1, T)
```

$$\frac{(z^2 - 1.755z + 1)}{(z-0.8187) (z^2 - 1.667z + 0.8187)}$$

Sampling time (seconds): 0.1

Add a gain, k, to make the DC gains match up:

```
DCs = evalfr(Gs,0)
```

```
73.5294
```

```
DCz = evalfr(Gz,1)
```

```
8.8913
```

```
k = DCs / DCz
```

```
8.2699
```

So, the discrete-time model for $G(s)$ is....

```
Gz = zpk([exp(z1*T), exp(z2*T)], [exp(p1*T), exp(p2*T), exp(p3*T)], k, T)
```

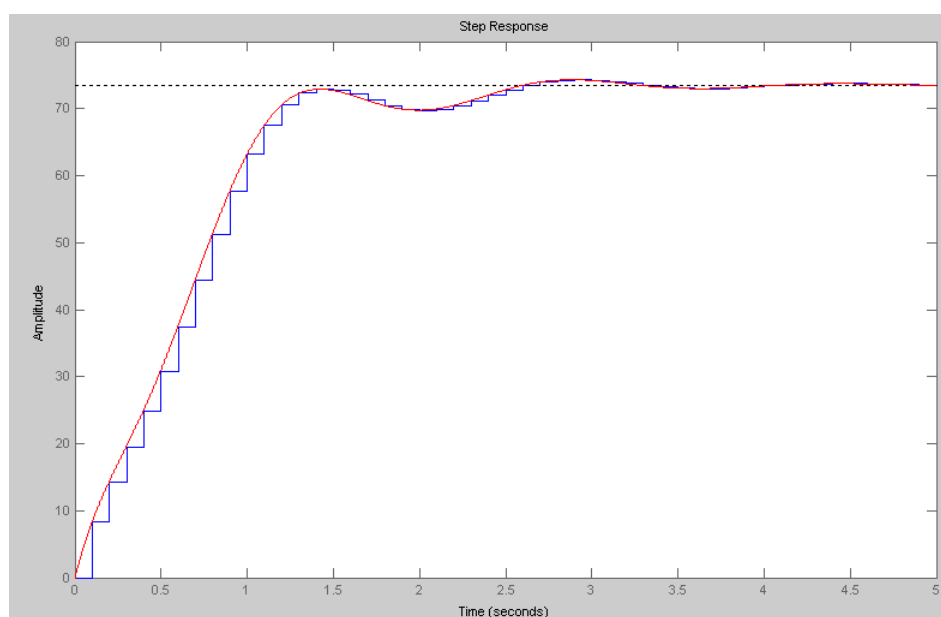
```
8.2699 (z^2 - 1.755z + 1)
```

```
-----  
(z-0.8187) (z^2 - 1.667z + 0.8187)
```

```
Sampling time (seconds): 0.1
```

Check the result by plotting the step response of $G(s)$ and $G(z)$ on the same graph:

```
step(Gz)
hold on
t = [0:0.001:5]';
ys = step(Gs,t);
plot(t,ys,'r');
```



Changing the Sampling Rate:

Note that the conversion from s to z depends upon the sampling rate. If you change the sampling rate, the filter $G(z)$ changes completely.

For example,

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

becomes the following with a sampling rate of $T = 0.1$:

$$G(z) \approx \left(\frac{0.051969 \cdot z}{(z-0.9048)(z-0.7408)(z-0.3687)} \right)$$

If you change the sampling rate to $T = 0.01$ (10ms), then

$$G(z) \approx \left(\frac{0.00009393 \cdot z}{(z-0.9900)(z-0.9704)(z-0.9048)} \right)$$

Changing the sampling rate is a big deal: it completely changes the system $G(z)$ and likewise will require a complete redesign of the compensator, $K(z)$.

Frequency Response of $G(z)$

Another way to compare filters is to compare the frequency response

If two filters have the same gain vs. frequency, they are the same filter.

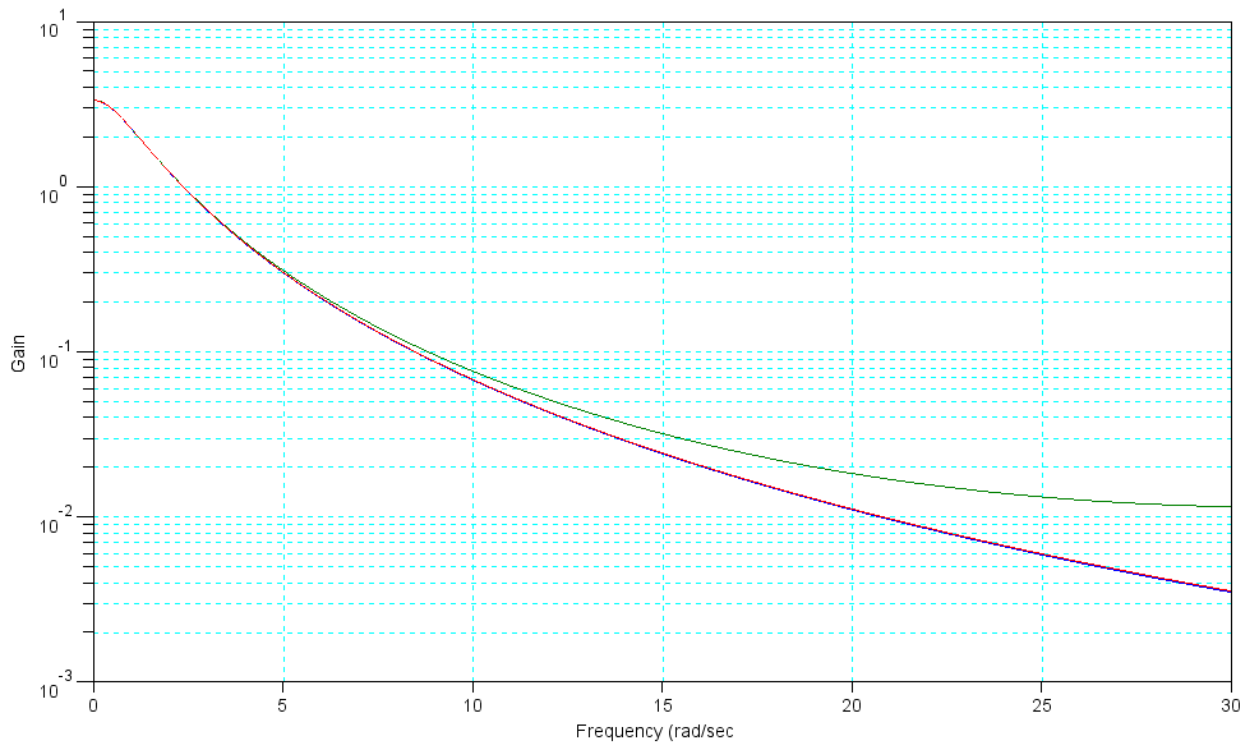
For example, plot the frequency response of the above three filters. In Matlab

```
w = [0:0.01:30]';
s = j*w;
Gs = 100 ./ ( (s+1) .* (s+3) .* (s+10) );

T = 0.1;
z = exp(s*T);
Gz = 0.051969*z ./ ( (z-0.9048) .* (z-0.7408) .* (z-0.3687) );

T = 0.01;
z = exp(s*T);
Gz2 = 0.000093938*z ./ ( (z-0.9900) .* (z-0.9704) .* (z-0.9048) );

plot(w, abs(Gs), w, abs(Gz), w, abs(Gz2));
xlabel('Frequency (rad/sec)');
ylabel('Gain');
xgrid(4)
```



Gain of $G(s)$ (blue), $G(z)$ with $T = 0.1$ (green), and $G(z)$ with $T = 0.01$ (red)