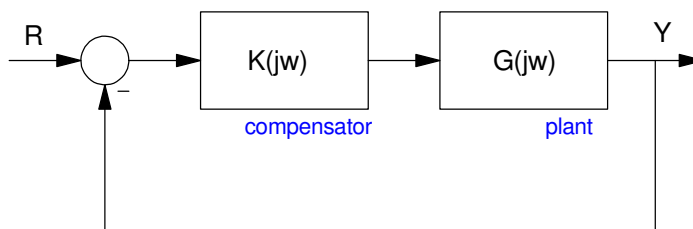


Bode Plots

Frequency Domain Techniques

Up to now, we've been looking at the time response for a closed-loop system. Another way of looking at a feedback system is from its frequency response. With this point of view

- $G(j\omega)$ is a filter with a gain and a phase shift
- $K(j\omega)$ is a pre-filter which warps the frequency response, turning a poor system, $G(j\omega)$, into a good system, GK



Frequency domain: Another way of looking at a feedback system

The origin of this approach dates back to the 1930's and the design of feedback amplifiers. Vacuum tube amplifiers (and transistor amplifiers) have a gain which varies with frequency: $G(j\omega)$. In order to make the gain flat (reducing the amplifier's distortion), feedback was introduced.

Sometimes, the feedback made a very good amplifier. Sometimes, the amplifier would just squawk and ring. The idea behind $K(j\omega)$ was to add a pre-filter to make a bad amplifier behave like a good amplifier.



Vacuum Tube Amplifier (Amazon.com, where else?)

From our standpoint, it doesn't really matter how you analyze a feedback control system: you should get the same result if you analyze in the time-domain using root-locus or in the frequency domain using techniques we'll shortly cover. Both should work.

For some systems, however, root locus works better. For other systems, frequency domain techniques work better. For completeness, we're covering both in this class.

As a sidelight, other faculty have noticed that whatever method you cover first in this class is the method students prefer. We covered root locus, so you'll probably like that method best. Other faculty cover frequency domain techniques first: their students tend to like these methods best. They both work.

Bode Plots

When we first started out in this class, we looked at

- Finding the step response given the transfer function, $G(s)$, and
- Finding the transfer function, $G(s)$, given the step response.

In this lecture, we look at

- Finding the frequency response given the transfer function, $G(s)$, and
- Finding $G(s)$ given the frequency response.

One problem with plotting the frequency response of $G(s)$ is there are three variables to plot on two-dimensional paper:

- The frequency, ω
- The gain, $|G(j\omega)|$, and
- The phase shift: $\angle G(j\omega)$

The way you plot this determines the name of the plot

Name	X-Axis	Y-Axis
Bode Plot	Frequency on a log scale	Gain in dB
Nichols Chart	Phase in degrees	Gain in dB
Nyquist Diagram	$\text{real}(G(j\omega))$	$\text{imag}(G(j\omega))$
Inverse Nyquist Diagram	$\text{real}(1/G(j\omega))$	$\text{imag}(1/G(j\omega))$

In this lecture, we're looking at Bode Plots.

First, as a definition, the gain of a system in dB (decibels) is

$$\text{gain} = |G(j\omega)|$$

$$\text{dB} = 20 \log_{10}(\text{gain})$$

$$\text{gain} = 10^{\text{dB}/20}$$

Determine the Frequency Response given $G(s)$

Two ways to do this are numerical methods (i.e. Matlab) and graphical methods. These are probably easiest to explain through an example:

Problem: Draw the Bode plot for

$$G(s) = \left(\frac{10s(s+100)}{(s+1)(s+10)} \right)$$

Numerical Solution (Matlab):

Substitute

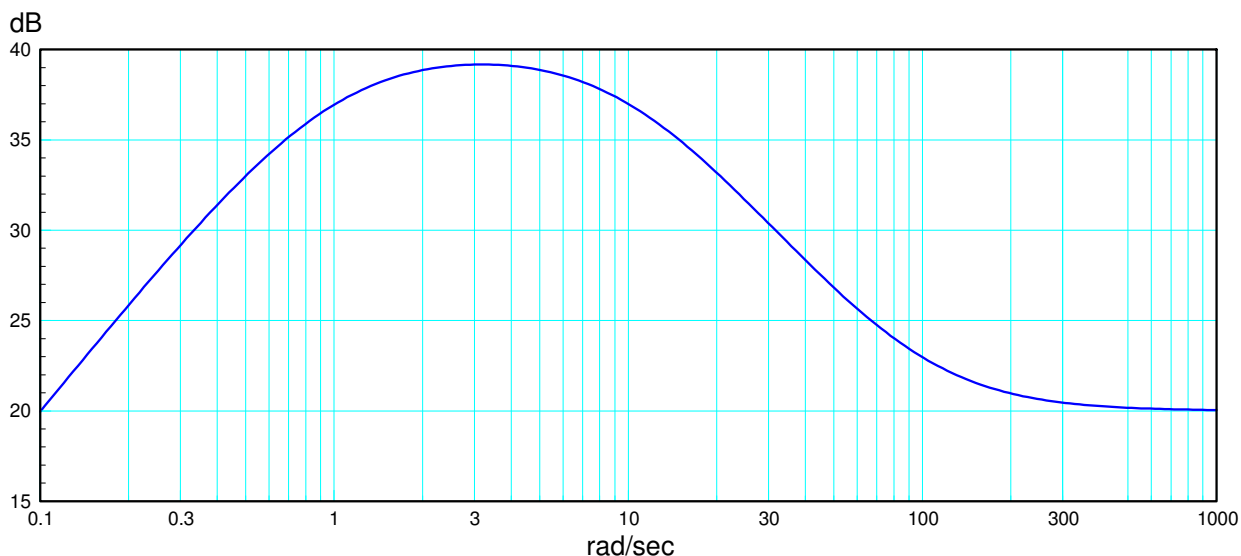
$$s \rightarrow j\omega$$

Calculate the gain at a bunch of points and plot as

- X axis: frequency on a log scale
- Y axis: gain in dB

In Matlab:

```
w = logspace(-1, 3, 250)';
s = j*w;
Gs = 10*s.*(s+100) ./ ( (s+1) .* (s+10) );
dB = 20*log10(abs(Gs));
semilogx(w, dB);
```



Bode plot for $G(s)$

Bode plots are really useful, so you can also write a Matlab m-file to do this

```
function [Gw ] = Bode2( G, w )
    Gw = 0*w;
    for i=1:length(w)
        Gw(i) = evalfr(G, j*w(i));
    end
    GdB = 20*log10(abs(Gw));
    semilogx(w, GdB);
end
```

This is called as

```
G = zpk([0,-100],[-1,-10],10);
w = logspace(-1,3,250)';
Gw = Bode2(G,w);
semilogx(w, 20*log10(abs(Gw)));
```

Note: Matlab has a function `Bode`, but this returns a 4-dimensional vector (!?). If `w` is a 250x1 vector, `G(jw)` should also be a 250x1 vector. `Bode2` does that.

Graphical Solution:

Back in the days of slide rules, complex numbers were painful to use. Likewise, the previous method which involves analyzing $G(s)$ for 250 different complex numbers was untenable. Instead, graphical techniques were developed. (These will be useful when we go backwards and try to find $G(s)$ given the frequency response).

First, note that the function

$$G(s) = s^n$$

plots as a straight line on a Bode plot. The gain is

$$\begin{aligned} dB &= 20 \log_{10}(|(j\omega)^n|) \\ &= 20n \log_{10}(\omega) \end{aligned}$$

which is a slope of 20ndB/decade (as ω goes up 10x, the gain goes up 20n dB)

Next, approximate

$$(s + a) \approx \begin{cases} a & |s| < |a| \\ s & |s| > |a| \end{cases}$$

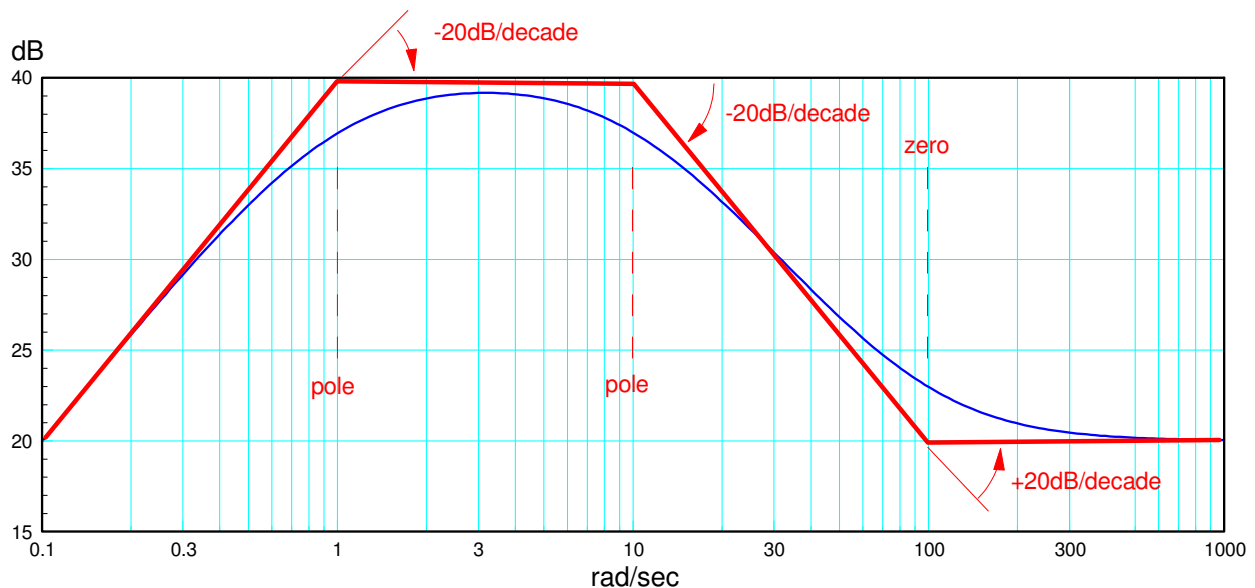
This gives

$$G(s) = \left(\frac{10s(s+100)}{(s+1)(s+10)} \right) \approx \begin{cases} \left(\frac{10s(100)}{(1)(10)} \right) = 100s & |s| < 1 \\ \left(\frac{10s(100)}{s(10)} \right) = 100 & 1 < |s| < 10 \\ \left(\frac{10s(100)}{(s)(s)} \right) = \frac{1000}{s} & 10 < |s| < 100 \\ \left(\frac{10s(s)}{(s)(s)} \right) = 10 & 100 < |s| \end{cases}$$

In each region, $G(s)$ is a straight line with a slope of $20n$ dB/decade where n is the power of s :

- Start at 20dB at $s = 0.1$
- The initial slope is +20dB/decade
- At 1 rad/sec, you encounter a pole. The slope drops 20dB/decade
- At 10 rad/sec, you encounter another pole. The slope drops another 20dB/decade
- At 100 rad/sec, you encounter a zero. The slope increases 20dB/decade

This gives the red lines in the Bode plot below. Note that each pole and zero produces a corner. This gives rise to the term 'corner frequency'. That simply means 'magnitude of the pole (or zero)'.



Graphical approximation for the gain vs. frequency (red line) vs. actual gain vs. frequency (blue line)

The actual gain vs. frequency (blue line) will be a smooth curve which goes from one asymptote (red line) to the next. At the corner, the gain is

- down 3dB for a pole, and
- up 3dB for a zero.

3dB comes from

$$|j + 1| = 1.414 = 3dB$$

whereas the Bode approximation would say

$$|j + 1| \approx 1$$

Bode Plots with Complex Poles

If you have complex poles, they will appear in complex conjugate pairs. This results in

- Having two poles with the same magnitude, meaning
- The slope changes by -40dB/decade when you encounter a complex pole

The gain at the corner tells you the damping ratio. Assume for convenience that the magnitude of the pole is 1:

$$G(s) = \left(\frac{1}{s^2 + 2\zeta s + 1} \right)$$

At $s = j1$ (the corner frequency)

$$|G(j)| = \left| \frac{1}{(j)^2 + j2\zeta + 1} \right| = \frac{1}{2\zeta}$$

For complex poles, the gain at the corner is $\frac{1}{2\zeta}$

For example, sketch the Bode plot for

$$G(s) = \left(\frac{1000s^2}{(s^2 + 1.4s + 1)(s^2 + 5s + 100)} \right)$$

In Matlab

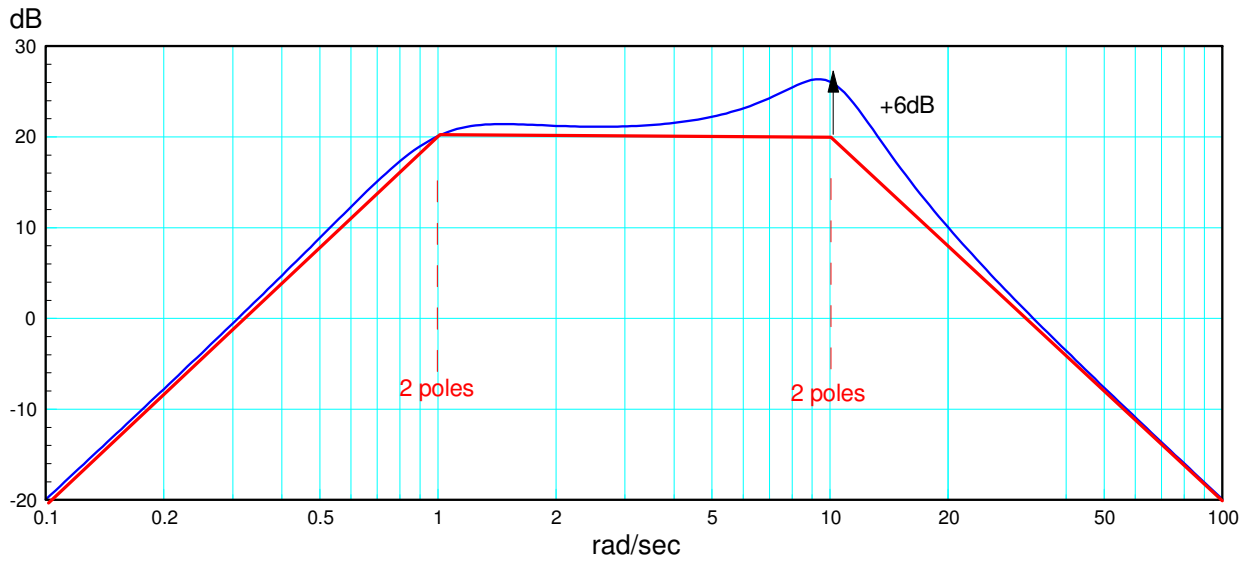
```
w = logspace(-1, 2, 250)';
s = j*w;
Gw = 1000*s.*s ./ ( ( s.^2 + 1.4*s + 1 ) .* (s.^2 + 5*s + 100) );
semilogx(w, 20*log10(abs(Gw)) );
```

Graphically,

- At 0.1 rad/sec, the gain starts out at -20dB (gain = 0.1)
- The slope increases 40dB / decade (s^2)
- At 1 rad/sec, you encounter two poles. The slope drops by 40dB/decade (net = 0dB/decade)
- At 10 rad/sec, you encounter two more poles. The slope drops by 40dB/decade (net = -40dB/decade)

At the corners

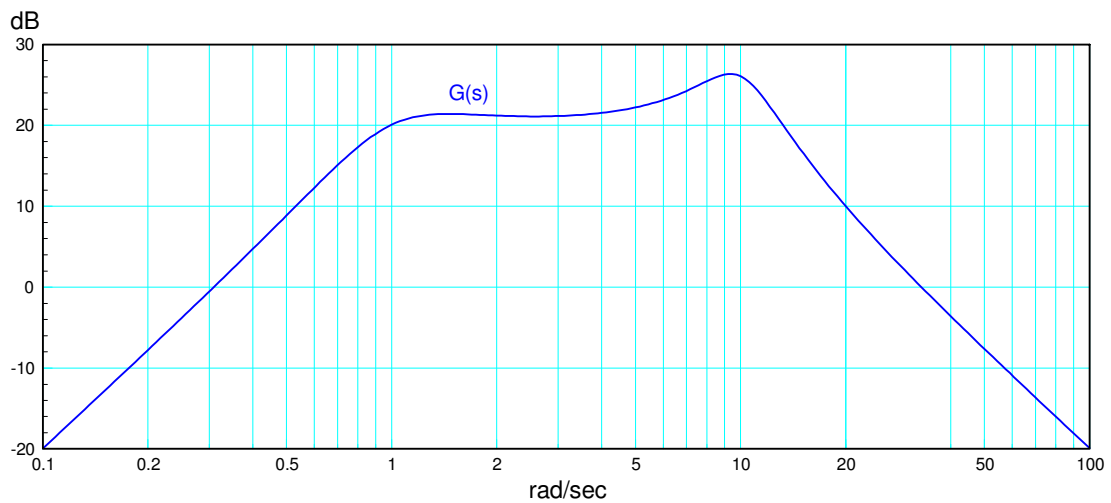
- At 1 rad/sec, the damping ratio is 0.5. The gain at the corner is 1 (0dB down from the corner)
- At 10 rad/sec, the damping ratio is 0.25. The gain at the corner is 2 (+6dB up from the corner)



Bode Plot (blue) and straight-line approximation (red)

Determining $G(s)$ from its Bode Plot

Assume you are given the frequency response (Bode plot) for $G(s)$

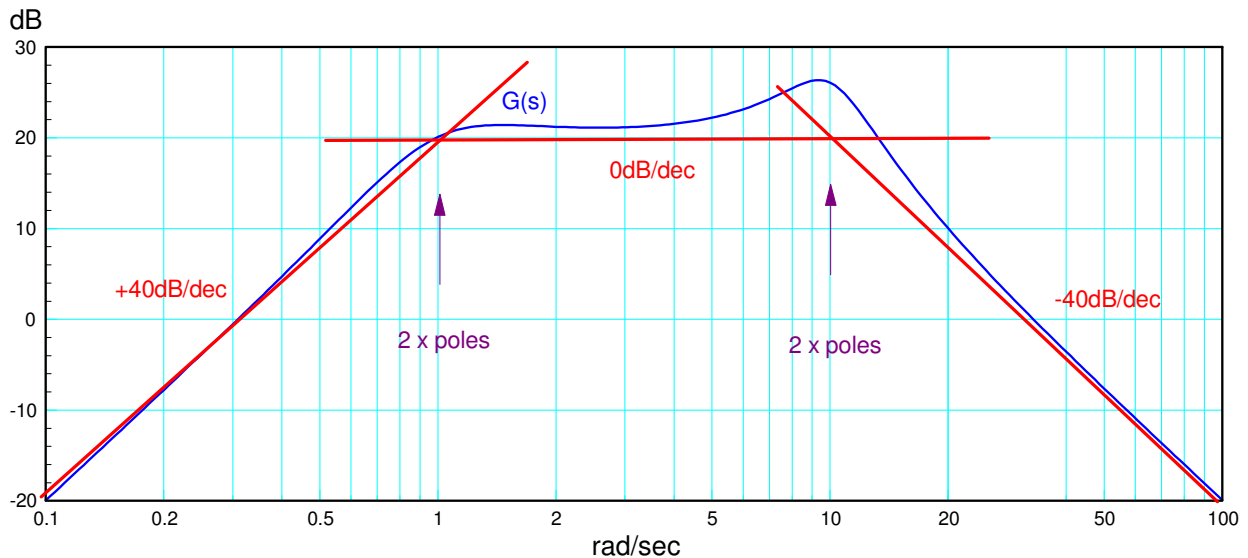


Given $G(j\omega)$, determine $G(s)$

The procedure for finding $G(s)$ is the reverse of the previous procedure.

Step 1: Draw in the asymptotes.

- Note: Each line *must* have a slope of $20n$ dB/decade
- 10dB/decade, for example, implies half of pole or zero, meaning a half derivative.



Step 1: Draw the asymptotes

At each corner, you have poles or zeros

- If the slope increased $20n$ dB/decade, you have n zeros at that frequency
- If the slope decreases $20n$ dB/decade, you have n poles at that frequency

For this graph, the slope starts out at +40dB/decade, telling you that there are two zeros left of 0.1 rad/sec (off the graph). Assume these are at $s = 0$

$$G(s) \approx \left(\frac{ks^2}{(s+1\angle\pm\theta)(s+10\angle\pm\phi)} \right)$$

Step 2: Determine the damping ratio of each pole

At 1 rad/sec, the gain at the corner is down 0dB (is passes through the corner)

$$\frac{1}{2\zeta} = 0dB = 1$$

$$\zeta = 0.5$$

$$\theta = \arccos(0.5) = 60^\circ$$

At 10 rad/sec, the gain at the corner is up +6dB (x2) relative to the corner, meaning

$$\frac{1}{2\zeta} = 2$$

$$\zeta = 0.25$$

$$\phi = \arccos(0.25) = 75.5^\circ$$

meaning

$$G(s) \approx \left(\frac{ks^2}{(s+1\angle\pm 60^\circ)(s+10\angle\pm 75.5^\circ)} \right)$$

Step 3: Solve for k:

The poles and zeros define the shape of the frequency response. 'k' slides the curve up and down.

Pick a point (any point) and match the gain there. For example

At 0.1 rad/sec, the gain is -20dB (0.1)

$$-20dB = 0.1 = \left| \frac{ks^2}{(s+1\angle\pm 60^\circ)(s+10\angle\pm 75.5^\circ)} \right|_{s=j0.1}$$

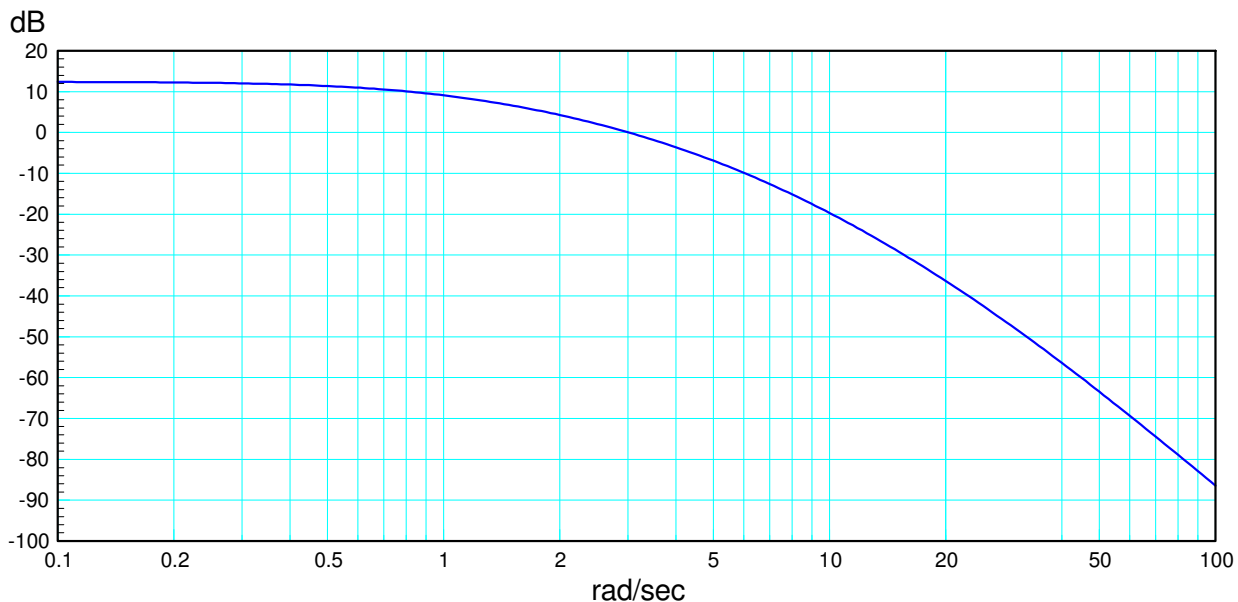
$$k = 9949.5$$

At 3 rad/sec, the gain is +22dB (12.59)

$$+22dB = 12.59 = \left| \frac{ks^2}{(s+1\angle\pm 60^\circ)(s+10\angle\pm 75.5^\circ)} \right|_{s=j3}$$

$$k = 1102.4$$

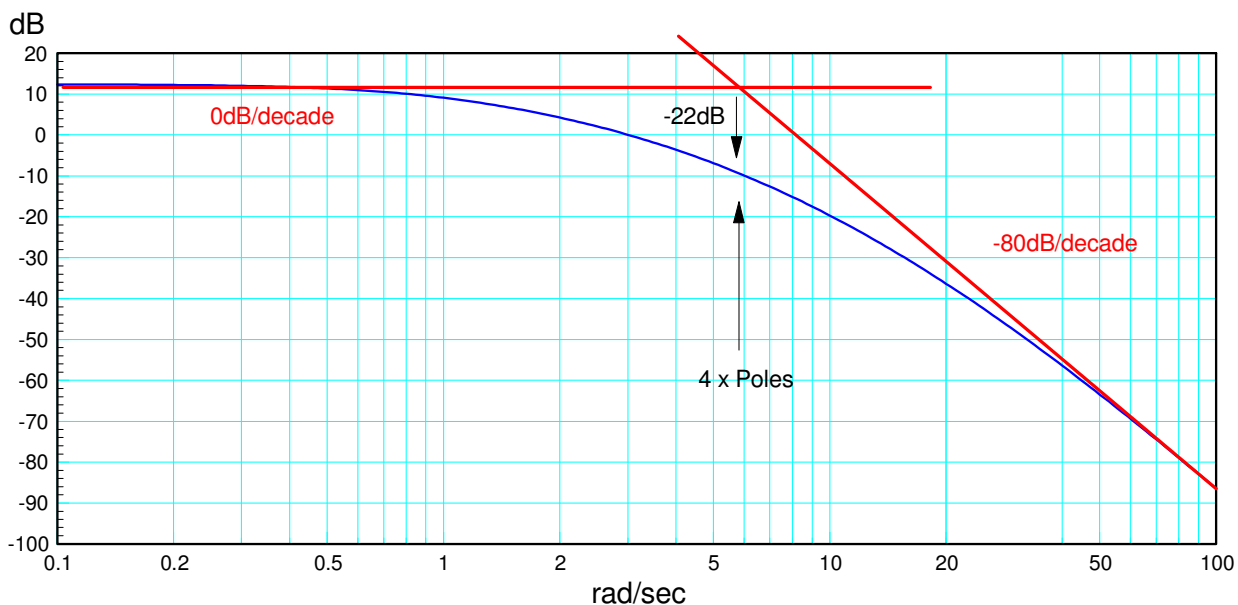
Example 2: Determine $G(s)$ given $G(j\omega)$



Example 2: Find $G(s)$

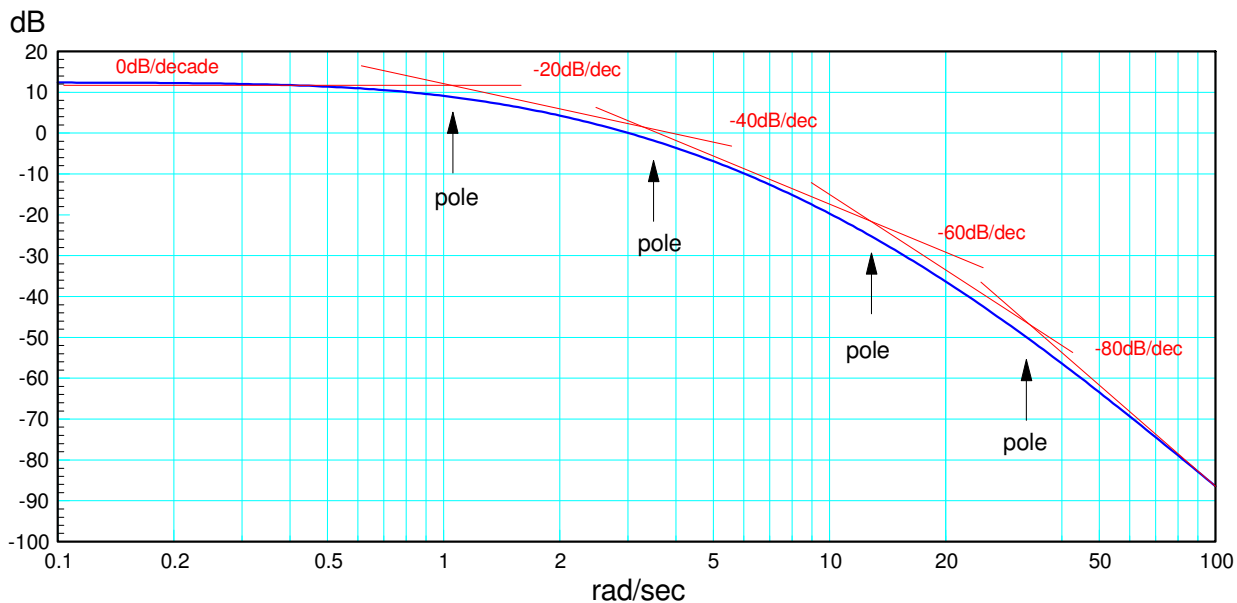
Step 1: Draw in the asymptotes. Start with two. Note that Two don't work since the gain at the corner is down 22dB

- Attributing -11dB to each pair of poles means the damping ratio is 1.77. You can't have a damping ratio more than one.



Approximating $G(s)$ with two asymptotes. The gain at the corner is down too much

What this means is you have real poles, not complex ones. Add in more asymptotes



Approximating G(s) with five asymptotes (meaning 4 real poles)

This results in

$$G(s) \approx \left(\frac{k}{(s+1.1)(s+3.3)(s+12)(s+30)} \right)$$

To find k, match the gain at some frequency

$$G(j0.1) = 12dB = 3.98 = \left(\frac{k}{(s+1.1)(s+3.3)(s+12)(s+30)} \right)_{s=j0.1}$$

$$k = 5225$$

resulting in

$$G(s) \approx \left(\frac{5225}{(s+1.1)(s+3.3)(s+12)(s+30)} \right)$$

Definitions

Open-Loop Gain:	$k G(j\omega)$
Closed-Loop Gain:	$\left(\frac{kG(j\omega)}{1+kG(j\omega)} \right)$
	dB 20 log(Gain)
0dB Gain Frequency:	Frequency where $ kG(j\omega) = 0\text{dB}$. Approximately equal to the bandwidth of the system. Approximately equal to the amplitude of the dominant pole (in rad/sec).
Gain Margin:	How far $G(j\omega)$ is away from -1 when the phase is 180 degrees. Used to determine stability.
Phase Margin	How far $G(j\omega)$ is away from -1 when the gain is one. Used as an approximation to resonance.
Resonance: (Mm)	The closest point from $G(j\omega)$ to -1 Determines the resonance.
M-Circle	Contour on a Nichols, Nyquist, or Inverse Nyquist plot where the closed-loop gain is a constant.
Nichols Chart:	Plot of $G(j\omega)$ in Cartesian coordinates. X = phase. Y = gain (dB). Graphical way to map open-loop gain to closed-loop gain Graphical way to determine the closest point to -1.
Nyquist Chart:	Plot of $G(j\omega)$ in Cartesian coordinates. X = real, Y = imag. Graphical way to map open-loop gain to closed-loop gain Graphical way to determine the closest point to -1.
Inverse Nyquist Chart	Plot of $1/G(j\omega)$ in Cartesian coordinates. X = real, Y = imag. Graphical way to map open-loop gain to closed-loop gain Graphical way to determine the closest point to -1.
Bode Plot	Two plots. i) Plot of Gain (dB) vs. log(frequency). ii) Plot of phase (degrees) vs. log(frequency).