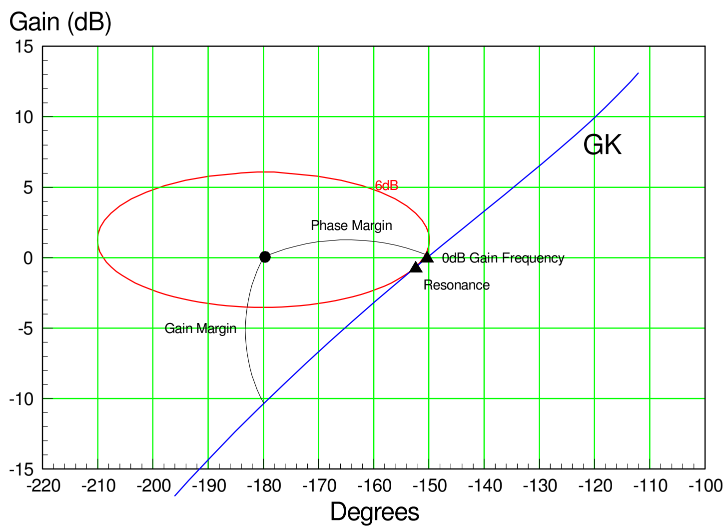
Compensator Design using Bode Plots

Gain Compensation

Nichols charts are useful since it shows directly what you are trying to do when designing a compensator: you are trying to keep away from -1 to limit the resonance. The problem with Nichols charts is

- They are difficult to draw
- The resonance is difficult to find: the frequency where G(jw) is tangent to the M-circle changes as you change the gain.

Note on the above Nichols chart, however, that the point where G(jw) is closest to -1 (and tangent to the M-circle) is very close to the point where the gain of GK = 0dB.



Nichols Chart along with the resonance (the closest point to -1), the phase margin, and the gain margin

If you assume that these two points are identical, it makes finding K much easier.

- i) Instead of drawing M-circles, find the point where $\left(\frac{G}{1+G}\right) = \left(\frac{1\angle\phi}{1+1\angle\phi}\right) = M_m$
- ii) Find the frequency where $\angle G = \phi$
- iii) Adjust K so that at this frequency, $|GK| = 1$

In this case, you are defining how far G(jw) is from -1 by how far it is away when the gain is 0dB. This is a much easier way to design a gain compensator, and hence has a special name: phase margin

Phase Margin: Distance of G(jw) to -1 when the gain of G(jw) = 1.

Relationship between Mm and Phase Margin:

The phase margin which approximately corresponds to a certain resonance is from

$G(j\omega) = 0dB\angle\phi = 1\angle\phi$

$\left|\frac{G}{1+G}\right| = \left|\frac{1\angle\phi}{1+1\angle\phi}\right| = M_m$

$|1 + (\cos\phi + j\sin\phi)| = \frac{1}{M_m}$

$(1+\cos\phi)^2 + (\sin\phi)^2 = \frac{1}{M_m^2}$

$1 + 2\cos\phi + \cos^2\phi + \sin^2\phi = \frac{1}{M_m^2}$

$$\boxed{2 + 2\cos\phi = \frac{1}{M_m^2}}$$

Phase Margin $= 180^0 - |\phi|$

Example: Find K so that $G(s) = \left(\frac{2000}{s(s+5)(s+20)}\right)$ has

- Mm < 6dB, and
- Minimal error for a step and ramp input.

Solution:

i) Find the phase margin corresponding to Mm = 6dB:

$\cos\phi = \left(\frac{\frac{1}{M_m^2}-2}{2}\right)$

$\phi = -151.04^0$

Phase Margin $= 28.96^0$

ii) Find the frequency where G(jw) has a phase shift of -151.04^0

$G(j5.2362) = 2.5474\angle -151.04^0$

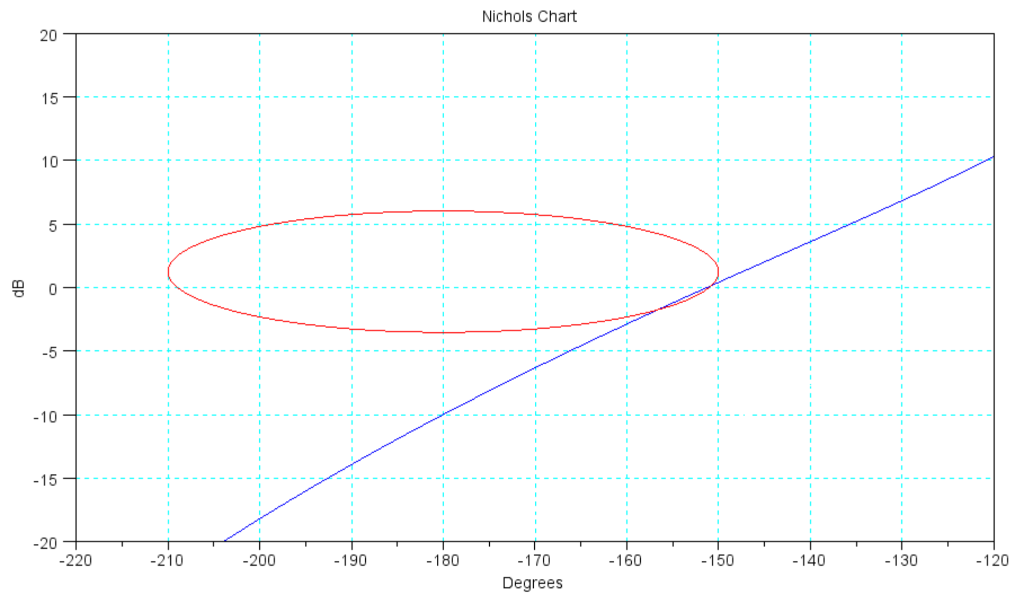
iii) Adjust k so that the gain at this frequency is one.

$k = \frac{1}{2.5474}$

$\boxed{k = 0.3926 = -8.1dB}$

With k=0.3926, the system will behave as

What happens can be seen on Nichols chart. With phase margins, we assume that G(jw) will be tangent to the M-circle at 0dB. While this is almost true, it actually intersects the M-circle at 0dB and passes slightly inside. This results in the gain we compute being a tad too large and the resulting resonance being slightly too high.



Nichols Chart for k*G(jw) with k selected for a 29 degree phase margin.
Note that kG(jw) intersects the M-circle at 0dB

This results in the following specifications for the closed-loop system will be approximately:

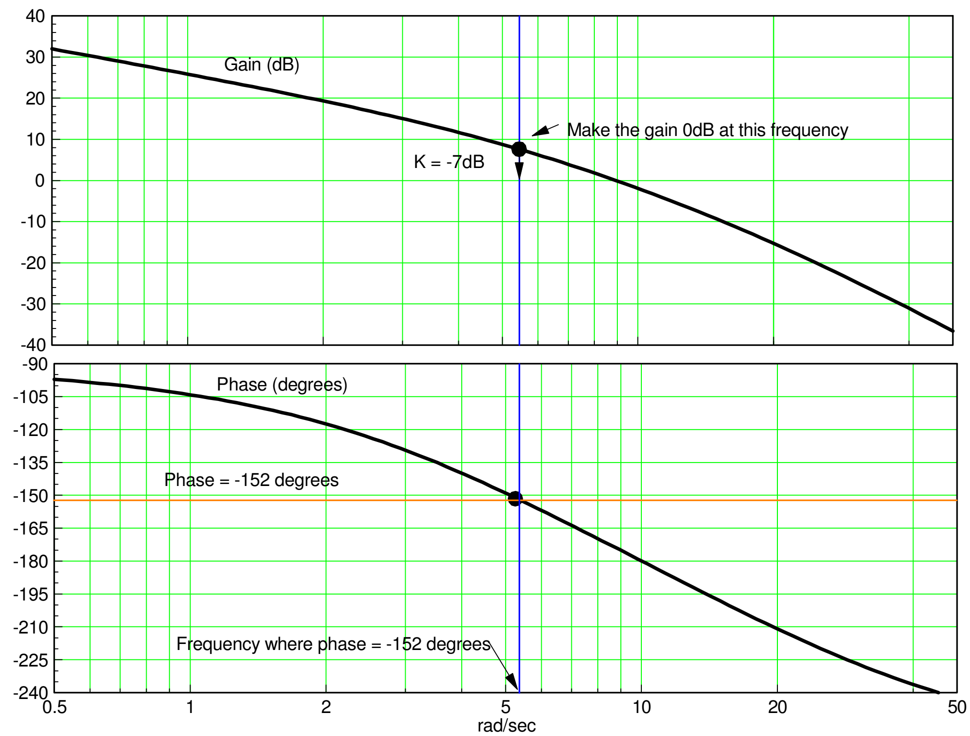
K(s)	Kv	0dB Gain Freq	Phase Margin	Mm	CL Dom Poles (approx)
0.3926	7.85	5.24 rad/sec	28.96 deg	2.00 (6dB)	-1.41 + j5.24

$$K_v = \lim_{s\to 0} (s\cdot G\cdot K) = \left(s\cdot\left(\frac{2000}{s(s+5)(s+20)}\right)\cdot 0.3926\right)_{s\to 0}$$

$$M_m \approx \left(\frac{1\angle -151.04^0}{1+1\angle -151.04^0}\right)$$ *assumes G(jw) intersects the M-circle on a Nichols chart at 0dB*

Closed-Loop Dominant Pole: the complex part is roughly the resonance which is roughly the 0dB gain frequency. The real part is found from the angle of the pole, which comes from the damping ratio, which compes from Mm.

Example 2: Repeat where G(s) is not given. Instead, assume that only the Bode Plot of G(s) is given:



i) Find the phase margin corresponding to Mm = 6dB:

$$\phi = -151.04^0$$

Phase Margin = 28.96^0

ii) Find the frequency where G(jw) has a phase shift of -152^0. This is shown on the above Bode plot.

w = 5.2 rad/sec

iii) Adjust k so that the gain at this frequency is one. This shift is shown on the above Bode plot as well.

G(j5.2) = 7dB

GK(j5.2) = 0dB

so

K = - 7dB

Note that the same data can be found from the graph as from the transfer function - although it is difficult to get four decimal places of accuracy from a graph.

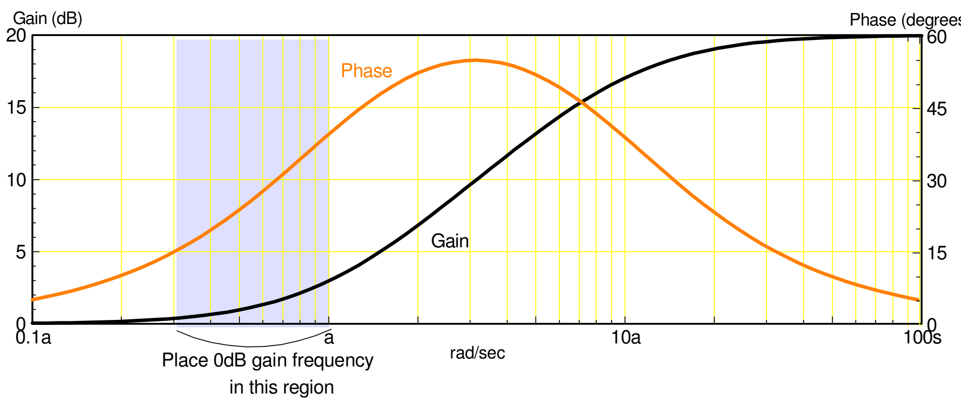
Lead Compensator Design using Bode Plots

The purpose of a lead compensator is to increase the phase margin. This implies that the lead compensator needs to add phase (pushing G(jw) to the right, away from -1 as shown on the Nichols chart but not gain (which would push G(jw) up towards -1).

The standard form for a lead compensator is

$$K(s) = 10\left(\frac{s+a}{s+10a}\right)$$

where the pole is 3 to 10 times larger than the zero (10 times here for convenience) and the DC gain is set to one. The gain and phase of K(s) is shown below.



Gain and Phase Shift of a Lead Copmensator

The first step in designing a lead compensator is determining how to pick the pole and zero, 'a'. In the previous example, G(s) has problems with phase at 4 rad/sec. At this frequency, you would like to add phase but not gain (i.e increase the phase margin of G). Note that

- If 'a' is too large (say a/10 = 4), you are adding minimal gain and minimal phase. The lead compensator will not help the phase margin much.
- If 'a' is too small (say 100a=4) you are adding gain but not phase. The added gain will push G(jw) closer to -1, making the system behave worse.
- If 'a' is just right (say a = (1..3)x4), you'll be adding a significant amount of phase without a lot of gain.

Since this is the purpose of a lead compensator

Rule: Pick the zero of the lead compensator to be 1 to 3 times the 0dB gain frequency of your system.

This should increase the phase margin. You can then use gain compensation to speed up the system.

Example: A gain compensator was designed for

$$G(s) = \left(\frac{2000}{s(s+5)(s+20)}\right)$$

so that the system had a 28.96 degree phase margin (meaning Mm = 2). This resulted in

K(s) = 0.3926

Resonance = 5.24 rad/sec

Add a lead compensator of the form

$$K(s) = 10\left(\frac{s+a}{s+10a}\right)$$

and find the system's phase margin with this lead compensator.

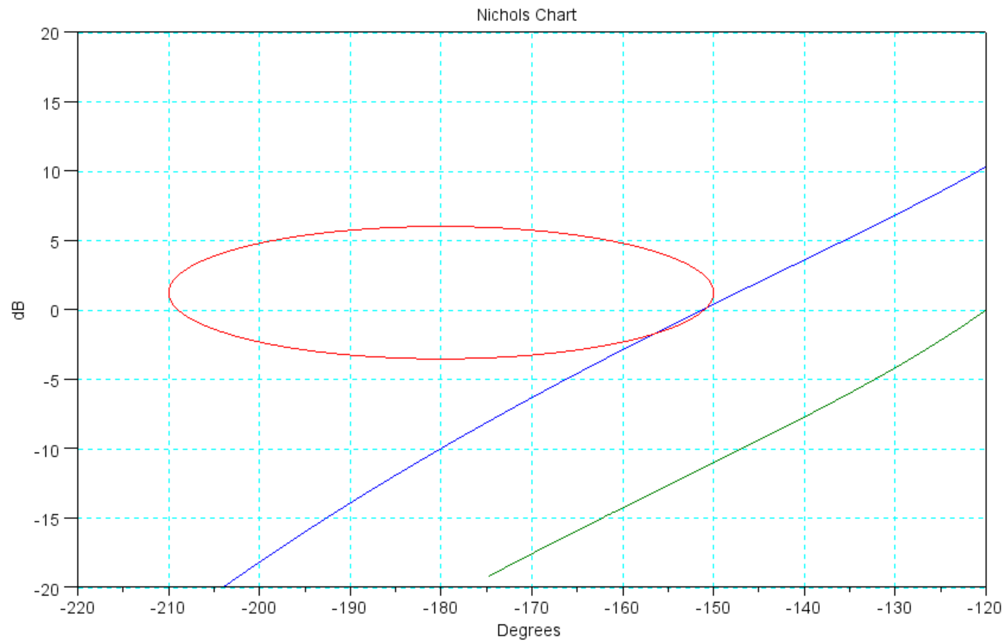
Solution: Step 1) Find the 0dB gain frequency. From before, this is 5.24 rad/sec.

Step 2) Pick the zero to be 1 to 3 times this frequency. Let a=6.

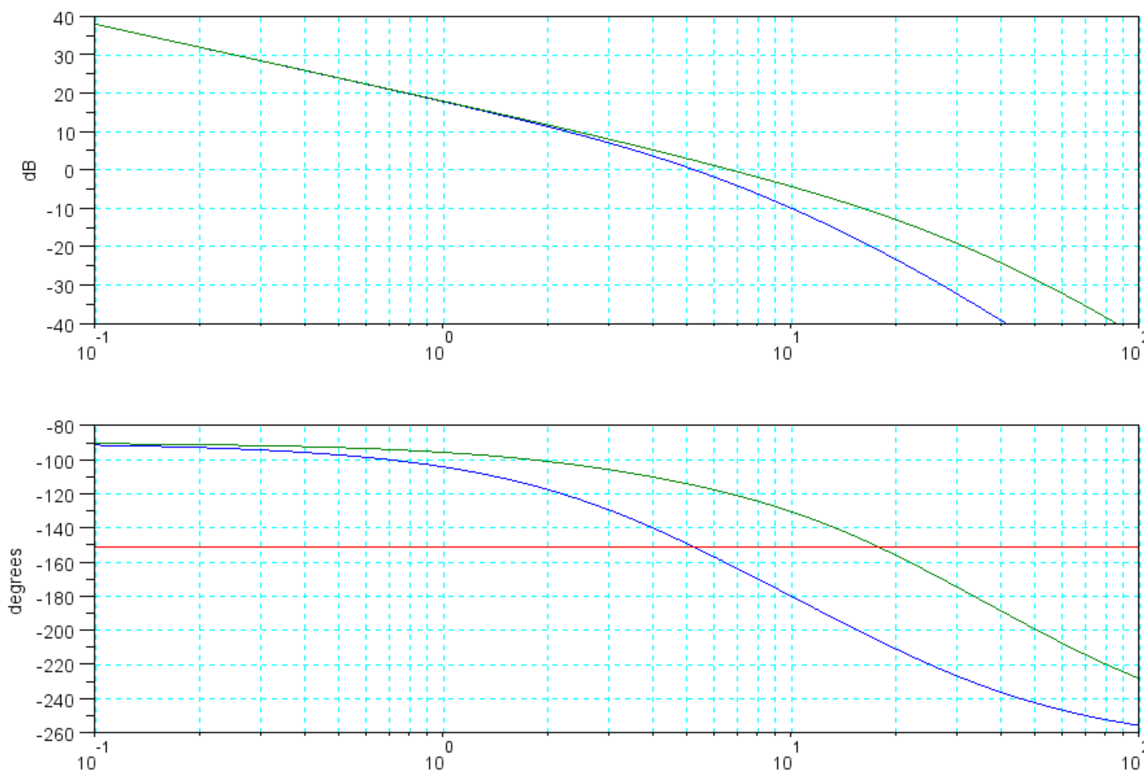
$$K_{gain}K_{lead} = (0.3929)\left(\frac{10(s+6)}{s+60}\right)$$

$$GK = \left(\frac{7852(s+6)}{s(s+5)(s+20)(s+60)}\right)$$

What the lead compensator does is it pushes G(jw) away from -1, increasing the phase margin. (i.e. it adds phase lead, hence the name *lead compensator*). This shows up on a Nichols Chart and a Bode Plot as follows:



Nichols Chart of G(jw) (blue line) showing a 29 degree phase margin (Mm = 2) and G*Lead (green line).
The lead compensator adds phase lead, pushing G(jw) to the right, away from -1.



Bode Plot of G(jw) (blue lines) showing a 29 degree phase margin (red line) at 5 rad/zed, and G*Lead (green line)
The lead compensator adds phase lead, pushing the phase curve up away from 180 degrees. This increases the phase margin.

With the lead compensator, the 0dB gain frequency is pushed out to 6.6446 rad/sec

$$GK(j6.6446) = 1.000\angle -119.82^0$$

This results in a 60.18 degree phase margin. This larger phase margin means the system is more stable / it will have a smaller resonance / the damping ratio is larger.

Step 3) Add more gain so that Mm = 2 (the phase margin is 29 degrees). At 17.7612 rad/sec,

$$K(s) = k\left(\frac{s+6}{s+60}\right)$$

$$GK = \left(\frac{2000(s+6)}{s(s+5)(s+20)(s+60)}\right)_{s=j17.7612} = 0.0684\angle -151.04^0$$

$$k = \frac{1}{0.0684} = 14.6292$$

so

$$K(s) = 14.6292\left(\frac{s+6}{s+60}\right)$$

This results in the following specifications for the closed-loop system:

K(s)	Kv	0dB Gain Freq	Phase Margin	Mm	CL Dom Poles (approx)
0.3926	7.85	5.24 rad/sec	28.96 deg	2.00 (6dB)	-1.41 + j5.24
$3.93\left(\frac{s+6}{s+60}\right)$	7.85	6.64 rad/sec	60.18 deg	1.00 (0dB)	-6.64 + j0
$14.93\left(\frac{s+6}{s+60}\right)$	29.86	17.76 rad/sec	28.96 deg	2.00 (6dB)	-4.78 + j17.76

Results for Gain, Lead, and Lead + Gain compensation

Note that lead + gain compensation (last row) gives you

- Better tracking (larger Kv)
- Wider bandwidth (larger 0dB gain frequency), and
- A faster system (which is the same thing as a wider bandwidth)

Lead Compensator Shortcut:

Note from the previous example that the system was

$$G(s) = \left(\frac{2000}{s(s+5)(s+20)}\right)$$

To speed up the system, a lead compensator was chosen in the form of

$$K(s) = k\left(\frac{s+a}{s+10a}\right)$$

where 'a' was 1 to 3 times the resonance you'd get with just gain comensation (5.2 rad/sec). This results in

$$5.2 < a < 15.6$$

From root locus techniques, you'd pick the zero to cancel the 2nd slowest pole (keep the pole at s=0 and cancel the next one), meaning

$$a = 5.$$

Not surprisingly, you get the same answer (give or take) as you'd get with root locus techniques. This lets us use a shortcut when designing a lead compensator when G(s) is given:

- Pick the zero to cancel the 2nd slowest pole
- If you're uncertain of the pole location, err on the high side.

From root locus, it isn't clear what happens if you miss the pole when cancelling it. From Bode plots, it's clear that it doesn't really matter if you miss the pole. On the previous gain vs. frequency for a lead compensator, nothing special happens at any particular frequency (the gain and phase are well-behaved smooth curves). Likewise, missing a pole is OK - you just have to get close.