LaPlace Transforms Review

ECE 461/661 Controls Systems

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Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & otherwise \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer functions and back.

Example 1: Find the transfer function that relates X and Y:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

The transfer function from X to Y is

$$\boxed{G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)}$$

Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Determine the differential equation that relates X and Y

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Note: Fractional powers are not allowed

- s^2Y means 'the second derivative of Y'.
- $s^{2.3}Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find y(t) given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X \qquad x(t) = 3\cos(4t)$$

Solution: This is a phasor problem (Circuits I)

$$X = 3 + j0 \qquad \text{real = cosine, imag = -sine}$$

$$s = j4$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j4}(3+j0) = -0.5435 - j0.9459$$

which means...

$$y(t) = -0.5435\cos(4t) + 0.9459\sin(4t)$$

Example 4: Find y(t) if

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X \qquad\qquad x(t) = 5\sin(20t)$$

Solution: Similar to before

$$X = 0 - j5$$

$$s = j20$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j20} (0 - j5) = -0.0230 + j0.0957$$

meaning

$$y(t) = -0.230\cos(20t) - 0.0957\sin(20t)$$

Note: Gain varies with frequency (it's a filter)

Example 5: Find y(t) if

$$x(t) = 3\cos(4t) + 5\sin(20t)$$

Solution: Use superposition. Treat this as two separate problems

- x(t) = 3 cos(4t)
- x(t) = 5 sin(20t)

The total input is the sum of the two x(t)'s.

The total output is the sum of the two y(t)'s

$$y(t) = -0.5435\cos(4t) + 0.9459\sin(4t)$$
$$-0.230\cos(20t) - 0.0957\sin(20t)$$

Solving Transfer Functions with Step Inputs

There are several ways to do this. My preference is to use a table

Common LaPlace Transforms		
Name	Time: y(t)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
decaying exponential	$a \cdot e^{-bt} u(t)$	$\frac{a}{s+b}$
damped sinusoid	$2a \cdot e^{-bt} \cos(ct - \theta) u(t)$	$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right)$

Example 6: Find the impulse response of G(s)

$$G(s) = \left(\frac{5}{s+3}\right)$$

Translating:

$$Y = \left(\frac{5}{s+3}\right)X \qquad x(t) = \delta(t)$$

Meaning

$$Y = \left(\frac{5}{s+3}\right)(1)$$

Using the table:

$$y(t) = 5e^{-3t}u(t)$$

Example 7: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Translating:

$$Y = \left(\frac{5}{s+3}\right)X \qquad\qquad x(t) = u(t)$$

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Not in the table, so use partial fraction expansion

$$\left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

Partial Fractions

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

Using the cover-up method

$$A = \left(\frac{5}{s+3}\right)_{s=0} = \frac{5}{3}$$

$$B = \left(\frac{5}{s}\right)_{s=-3} = -\frac{5}{3}$$

$$Y = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

meaning

$$y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)$$

Repeated Roots

Option 1: Use a table that includes repeated roots

Option 2: Change the problem

- Change it one that is easier to solve (no repeated roots)
- But keep the flavor of the original problem

$$G(s) = \left(\frac{1}{(s+1)(s+1)}\right) \approx \left(\frac{1}{(s+0.99)(s+1.01)}\right)$$

You'll have a hard time telling the difference

Solving with Complex Roots:

- If you don't mind complex numbers, complex roots are no harder than real roots
- Use the table entery:

$$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a \cdot e^{-bt}\cos(ct-\theta)u(t)$$

Example: Find the step resposne of

$$\left(\frac{15}{s^2+2s+10}\right) = \left(\frac{15}{(s+1+j3)(s+1-j3)}\right)$$

Solution:

$$Y = \left(\frac{15}{(s+1+j3)(s+1-j3)}\right)\left(\frac{1}{s}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+1+j3}\right) + \left(\frac{C}{s+1-j3}\right)$$

Use partial fractions

$$Y = \left(\frac{15}{(s+1+j3)(s+1-j3)}\right)\left(\frac{1}{s}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+1+j3}\right) + \left(\frac{C}{s+1-j3}\right)$$

Using the cover-up method

$$A = \left(\frac{15}{(s+1+j3)(s+1-j3)}\right)_{s=0} = 1.5$$

$$B = \left(\frac{15}{s(s+1+j3)}\right)_{s=-1-j3} = 0.7906\angle -161.5^0$$

$$C = \left(\frac{15}{s(s+1-j3)}\right)_{s=-1+j3} = 0.7906\angle +161.5^0$$

meaning

$$Y = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle -161.5^0}{s+1+j3}\right) + \left(\frac{0.7906\angle 161.5^0}{s+1-j3}\right)$$

$$y(t) = (1.5 + 1.5812\, e^{-t} \cos(3t + 161.5^0))u(t)$$

Controls Systems vs. Signals & Systems

Controls Systems is easier:

- This class uses single-sided, one-dimensional LaPlace transforms
  - Time is unidirectional (single-sided)
  - Time has one dimension

In Signals & Systems

- You can have non-causal filters (left & right for an image)
- You can have multiple dimension (X, Y, Z)
- You can have a complex number without its complex conjugate
  - Single Side-Band radio transmission

Matlab to the Rescue!

- Calculating step responses is really tedious.
- Matlab makes life *much* easier

Input a system:

Matlab Command	Meaning
`G = tf([2,3,4], [5,6,7,8])`	$G(s) = \left(\frac{2s^2+3s+4}{5s^3+6s^2+7s+8}\right)$
`G = zpk([-1,-2],[-3,-4,-5],10)`	$G(s) = \left(\frac{10(s+1)(s+2)}{(s+3)(s+4)(s+5)}\right)$
`G = ss(A,B,C,D)`	$sX = AX + BU$ $Y = CX + DU$

Step and impulse response

```
t = [0:0.01:10]';
y = impulse(G,t);
plot(t,y)
```

impulse response

```
t = [0:0.01:10]';
y = step(G,t);
plot(t,y)
```

step response

Combining Systems

```
A = zpk([],[-1,-2],10)
```

$$A = \left(\frac{10}{(s+1)(s+2)}\right)$$

```
B = tf(10,[1,6,4])
```

$$B = \left(\frac{10}{s^2+6s+4}\right)$$

```
G = minreal(A*B)
```

$$G = AB = \left(\frac{10}{(s+1)(s+2)}\right)\left(\frac{10}{s^2+6s+4}\right)$$

```
G = minreal(A + B)
```

$$G = A + B = \left(\frac{10}{(s+1)(s+2)}\right) + \left(\frac{10}{s^2+6s+4}\right)$$

Partial Fractions in Matlab

$$Y = \left(\frac{20(s+3)}{s(s+5)(s+10)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+5}\right) + \left(\frac{C}{s+10}\right)$$

```
>> G = zpk([-3],[0,-5,-10],20)
>> s = 0 + 1e-9;
>> A = evalfr(G,s) * (s+0)
     A = 1.2000


>> s = -5 + 1e-9;
>> B = evalfr(G,s) * (s+5)
     B = 1.6000


>> s = -10 + 1e-9;
>> C = evalfr(G,s) * (s+10)
     C = -2.8000
```