Root Locus

ECE 461/661 Controls Systems

Jake Glower - Lecture #20

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Routh Criteria

- Tells you if a system is stable
- Tells you the range of k for stability
- Doesn't tell you if a system has a good behavior

Example: $G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$

Routh Table result:

- $0 < k < 2.5$

What value do I pick?

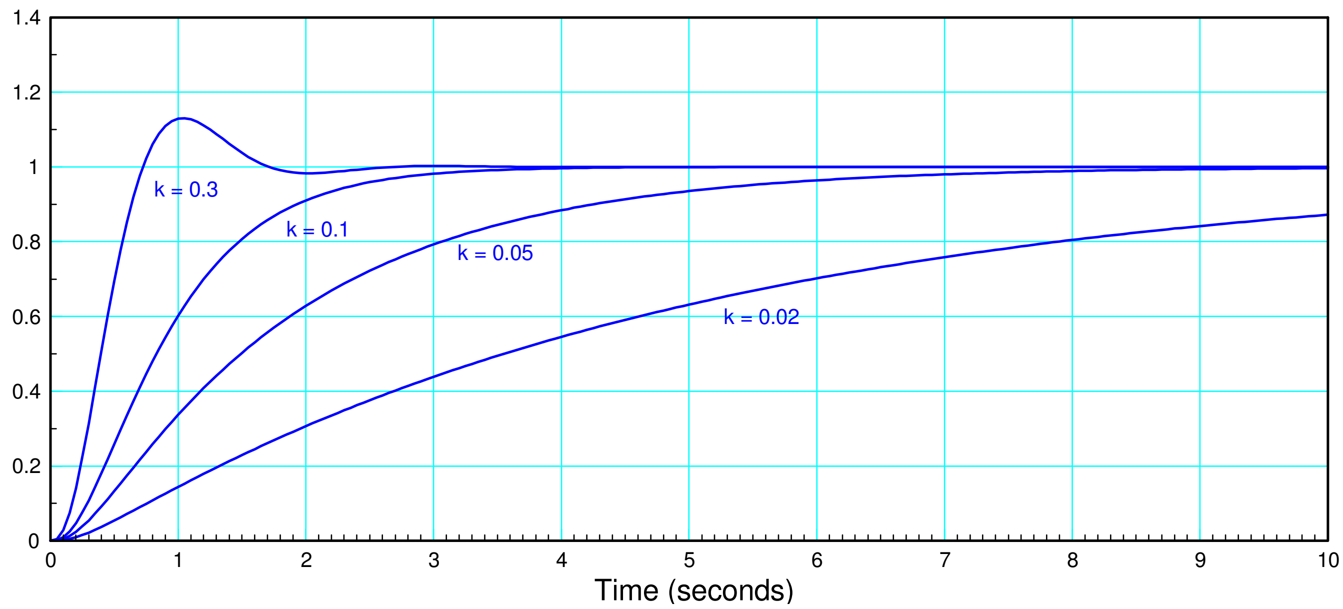
- What value is "best"?

What gain is best? $G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$

- Step response of $\left(\frac{Gk}{1+Gk}\right)$

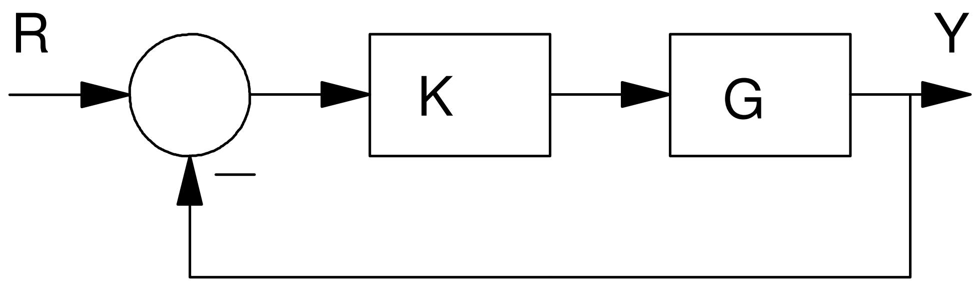
Note: Closed-Loop poles are shifting

- Large gains are good: faster response, less error
- Too much gain is bad: too much overshoot
- What is the largest gain you can use without getting too much overshoot?

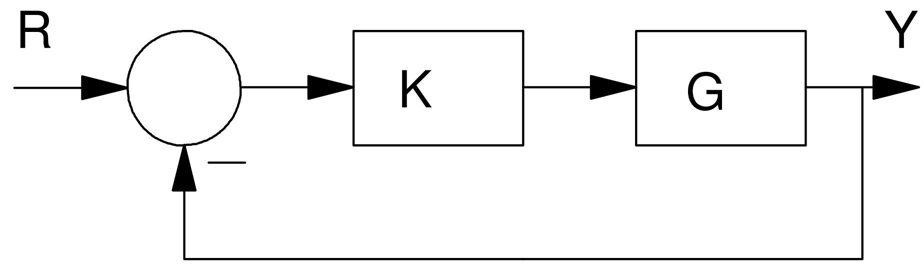


Root Locus

- The locus of the closed-loop roots for $0 < k < \infty$
- Gives you a shopping list: all responses possible
- "Best" gain (k) is the response with
  - the highest gain
  - that does't have too much overshoot



Definitions:



Open-Loop Gain: GK

Closed-Loop Gain: $\left(\frac{kG}{1+kG}\right)$

Open-Loop Poles: solutions to p(s)
= 0

Open-Loop Zeros: solutions to z(s) = 0.

Closed-Loop Poles: solutions to p(s) + kz(s) = 0.

Root Locus: Plot of the closed-loop poles as k varies from 0 to infinity.

Root Locus Plot: Method #1 (Brute Force)

- Pick 1000 values of k
- Calculate the roots for each k
- Plot the roots: real vs. complex

Example: Plot the root locus of

$$s(s+5)(s+20)+1000k=0$$

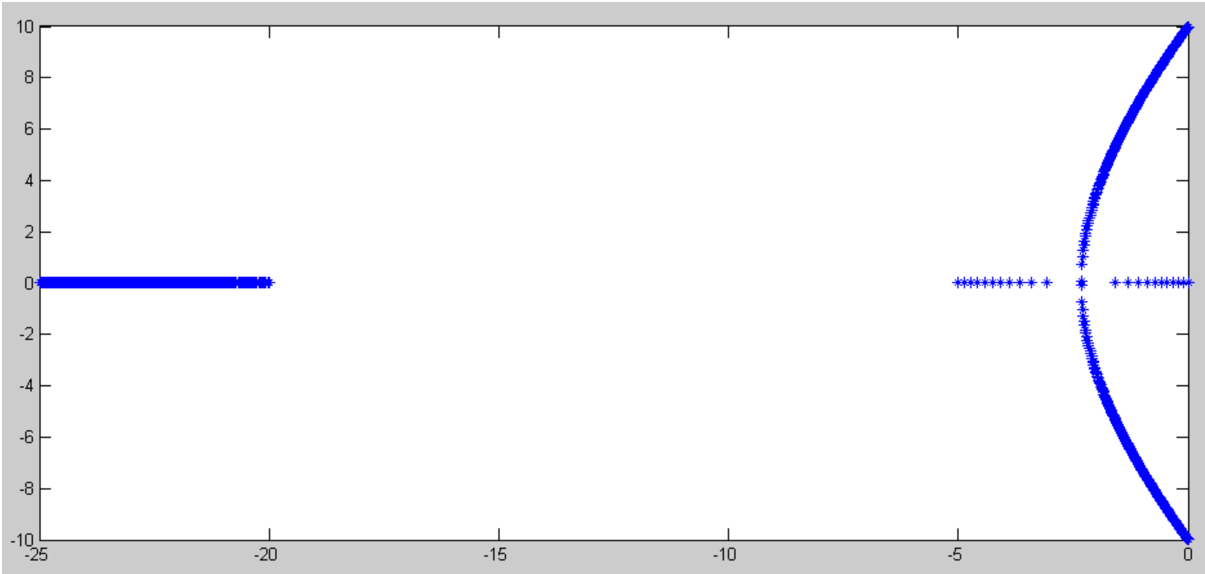
Solution: In Matlab

```
  p = poly([0,-5,-20]);

  z = [0,0,0,1000];

  k = [0:0.01:2.5]';
  for i=1:length(k)
    R = roots(p + k(i)*z);

 plot(real(R),imag(R),'*');
    hold on
    end
```



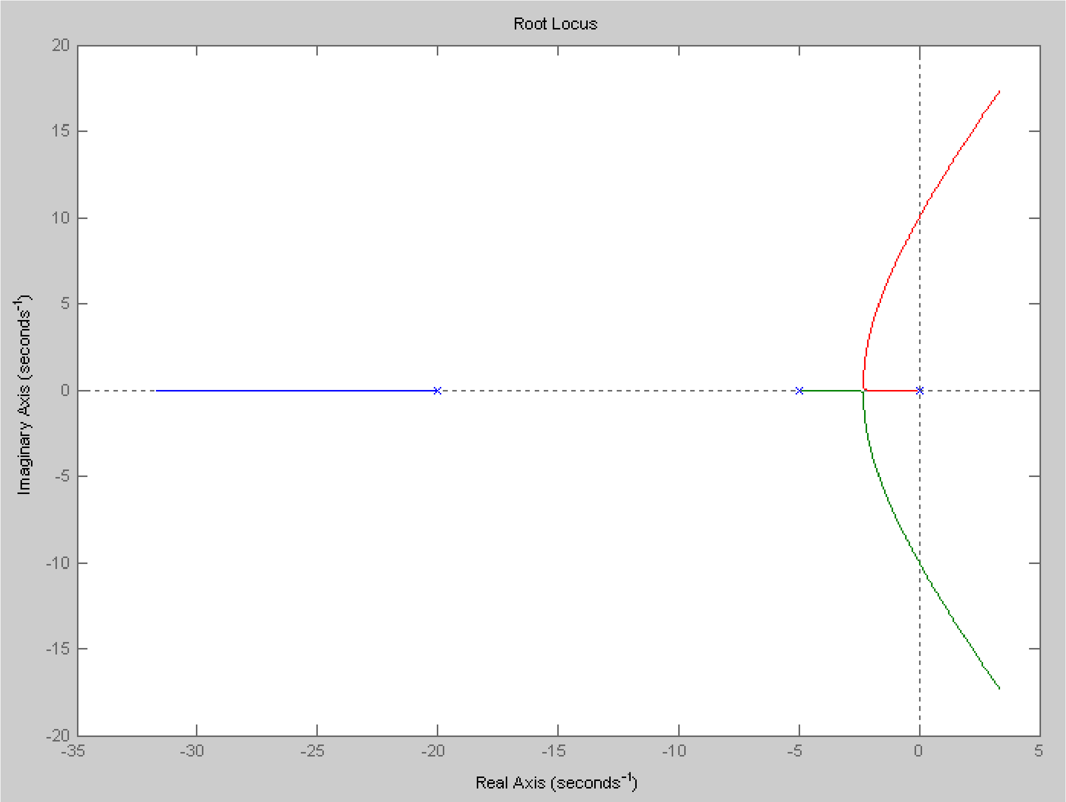
Method #2: Matlab Function *rlocus*

Actually the same as method #1

```
R = rlocus(G, k);
```

Example:

```
G = zpk([],[0,-5,-20],1000);
k = logspace(-2,1,1000)';
rlocus(G,k);
```

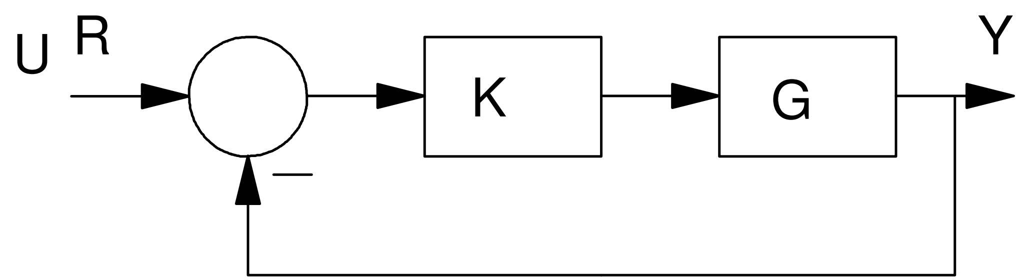


Method #3: Graphical Methods

- Better at explaining how the root locus behaves
- Better for designing compensators to improve the root locus

Procedure: Assume a unity feedback system

$$KG = \left(\frac{k\,z(s)}{p(s)}\right) = k\,\frac{\text{zeros}}{\text{poles}}$$



The closed-loop system is

$$\left(\frac{kG}{1+kG}\right) = \left(\frac{k\cdot z(s)}{p(s)+k\cdot z(s)}\right)$$

The closed-loop poles are

$$p(s) + k \cdot z(s) = 0$$

Rewrite this as

$$\frac{p(s)}{z(s)} = -k = k\angle 180^0$$

1) The amplitude of both sides must match, meaning

$$\boxed{k = \frac{\Pi(\text{distances from the poles to point s})}{\Pi(\text{distances from the zeros to point s})}}$$

or

$$|Gk|_s = 1$$

2) The angles must match:

$$\boxed{180^0 = \sum(\text{angles from the poles to point s}) - \sum(\text{angles from the zeros to point s})}$$

Root Locus Plots:

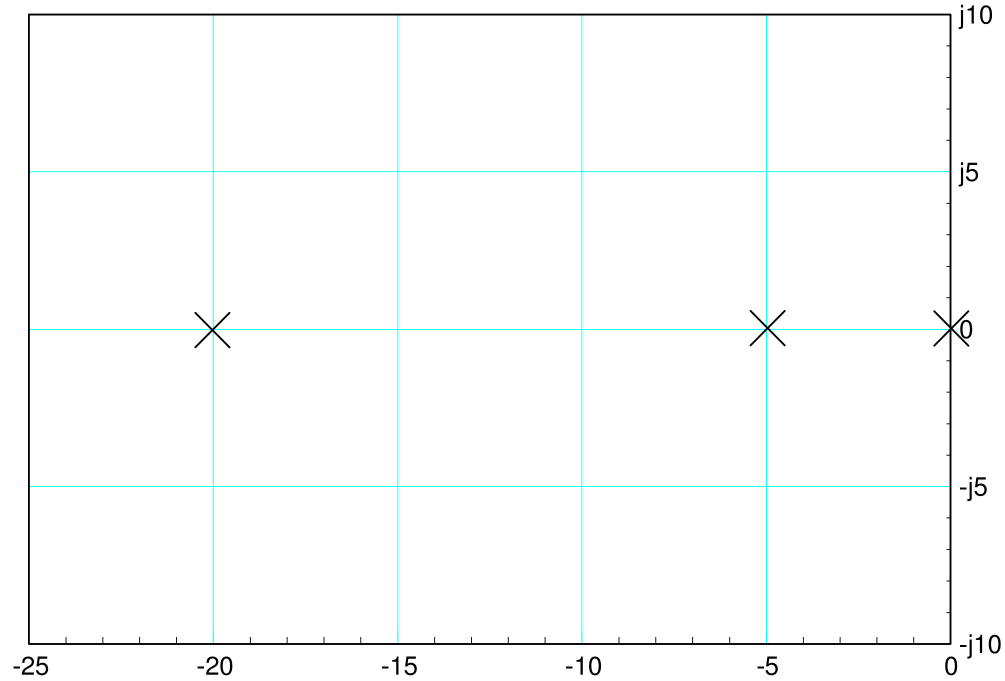
All points on the s-plane where

$$\frac{\angle p(s)}{\angle z(s)} = 180^0$$

Roots to

$$p(s) + k \cdot z(s) = 0$$

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$$



Poles:

At k=0, the roots start at the open-loop poles.

- Mark an 'X' at these points.

Zeros:

As $k \to \infty$, the roots go the open-loop zeros

- Mark an 'O' at these points.

Real Axis Loci:

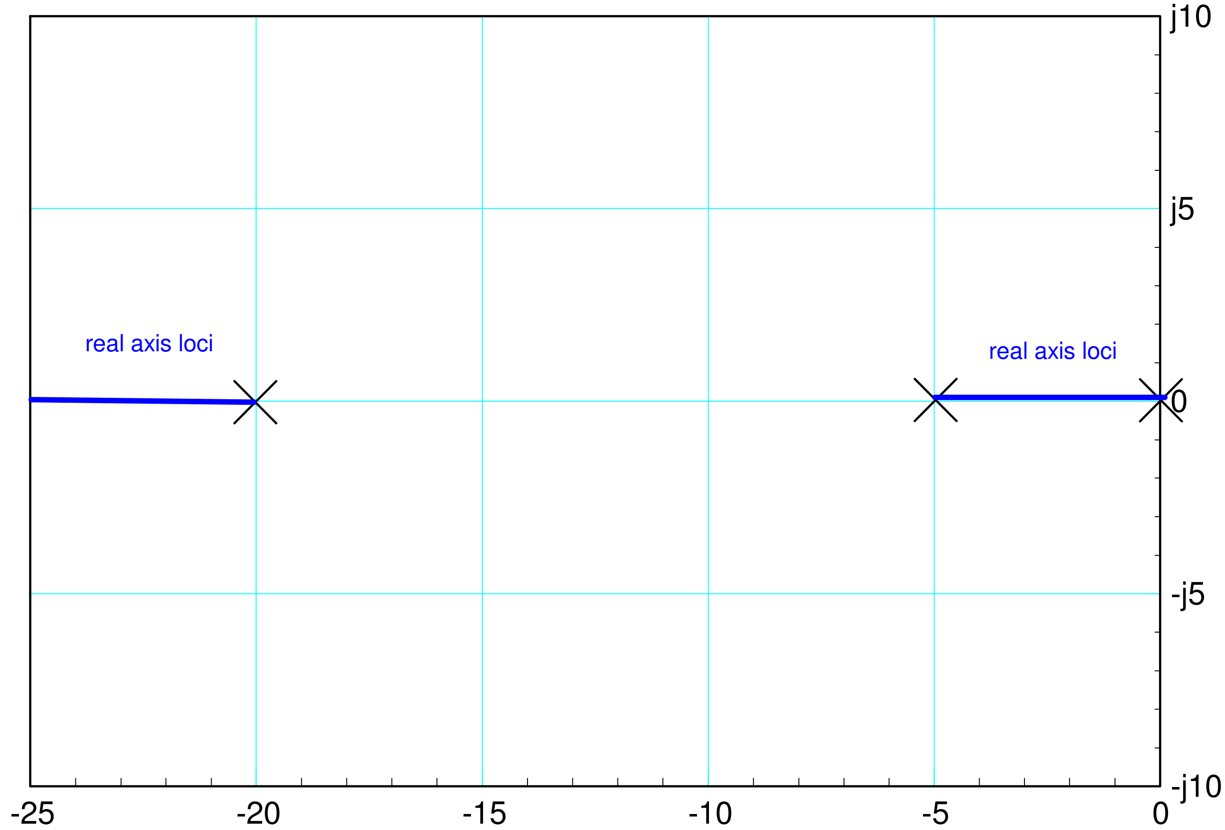
Angle from poles + Angle from zeros = 180 degrees

Works when there are an odd number

of poles & zeros

to the right

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$$



Real Axis Loci:

- (0, -5), (-20, -infinity)

Asymptotes

Number of Asymptotes:

- # poles - # zeros
- Poles have to go somewhere
- If not a zero, then an asymptote

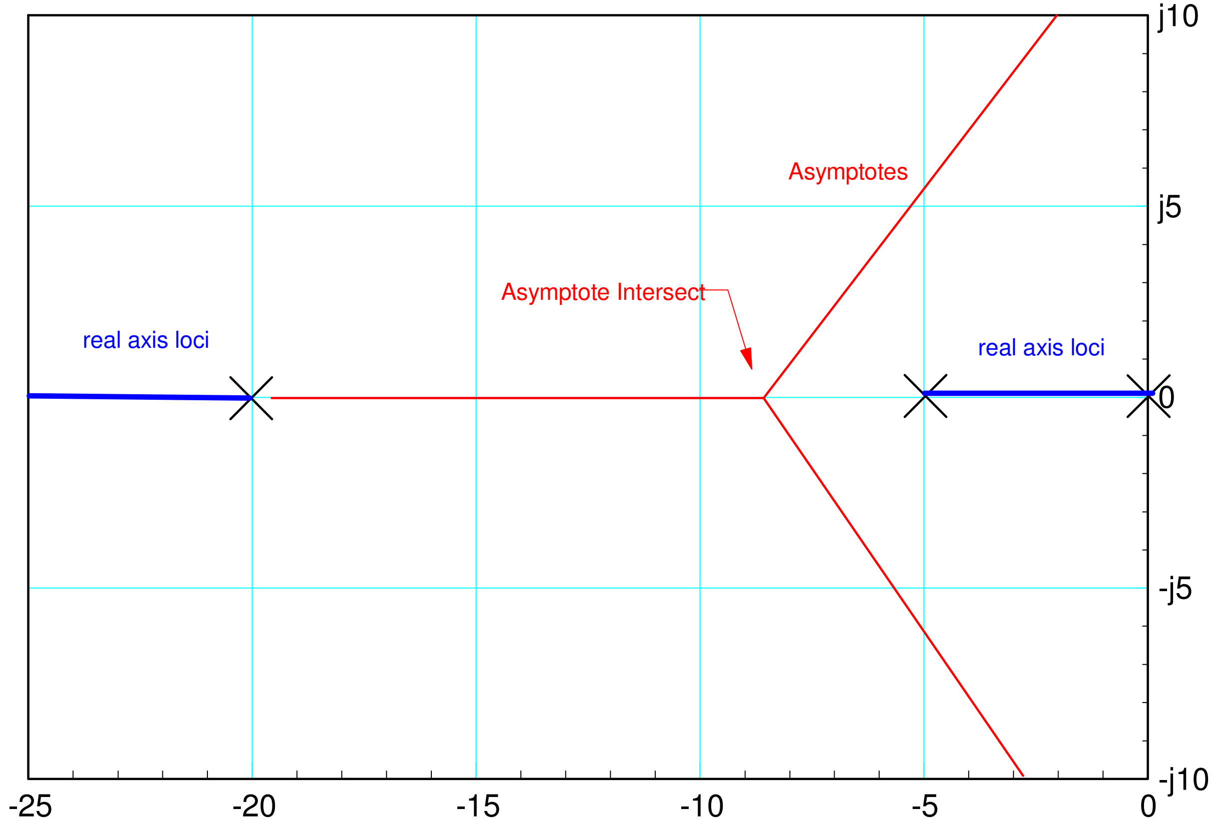
$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$$

Asymptote Angle

$$(n-m)\phi = 180^0$$

Asymptote Intersect

- Center of mass
- $\left(\frac{\sum \text{poles} - \sum \text{zeros}}{\#\text{poles} - \#\text{zeros}}\right)$



Breakaway Point:

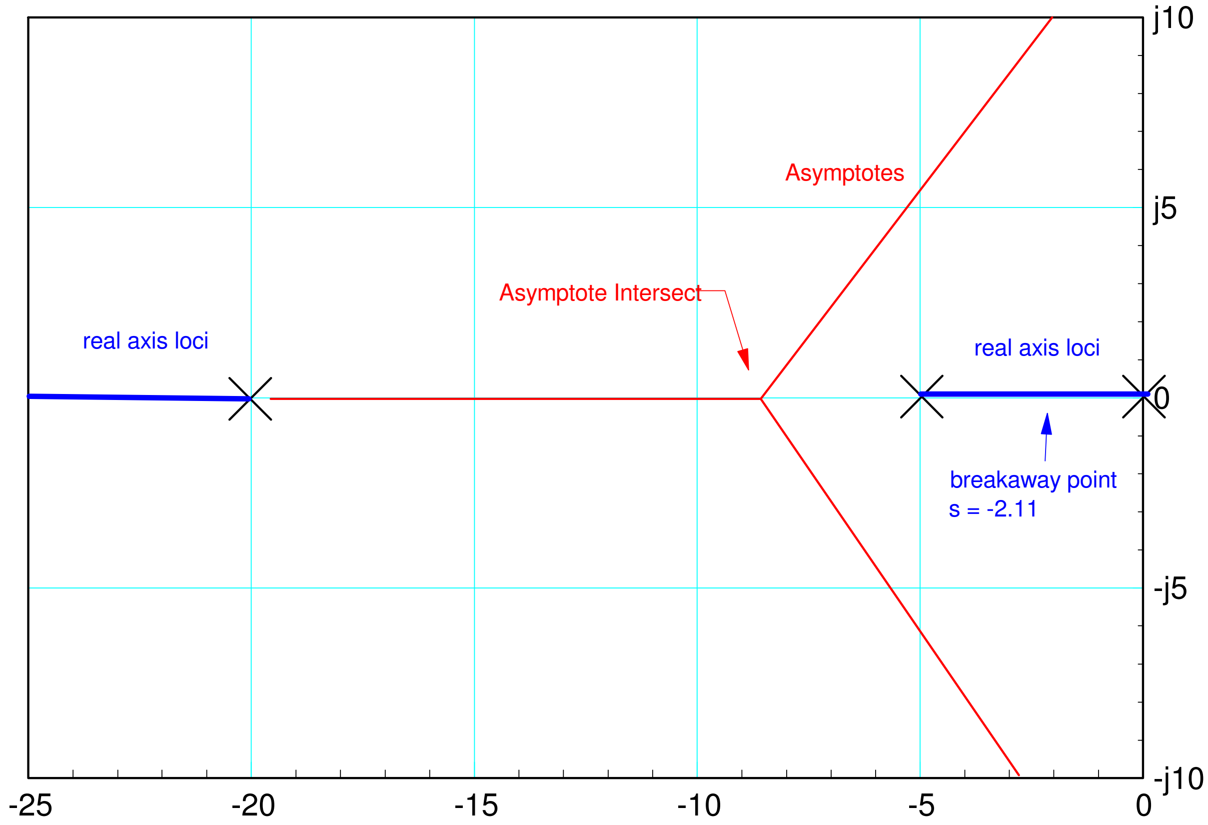
Where the poles break away from the real axis

- Equilibrium for an electron
  - Poles: minus charges
  - Zeros: plus charges
- Sensitivity is infinity:

$$\frac{d}{ds}\left(\frac{p(s)}{z(s)}\right) = 0$$

- Search along s = X + j0.01
  - angles add up to 180 degrees

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$$



jw Crossings:

Where the root locus crosses the jw axis

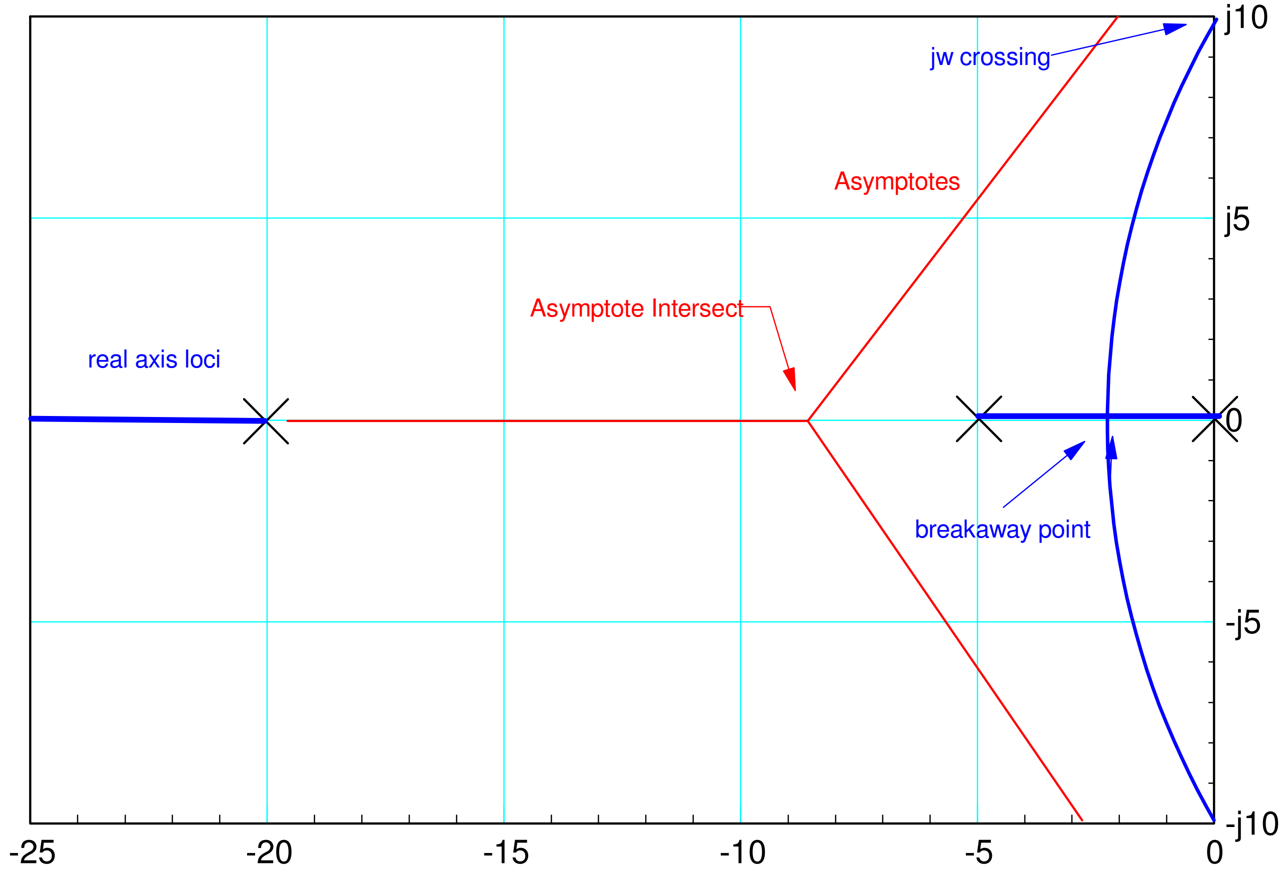
- Defines the minimum / maximum gain for stability

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)$$

$$angle(G(s))_{s=j\omega} = 180^0$$

No closed-form solution

- Iterate until angles add up



Departure Angle:

If a pole or zero is complex, the departure angle is the angle at which the root locus leaves the pole or enters the zero. This is found by summing the angles to 180 degrees.

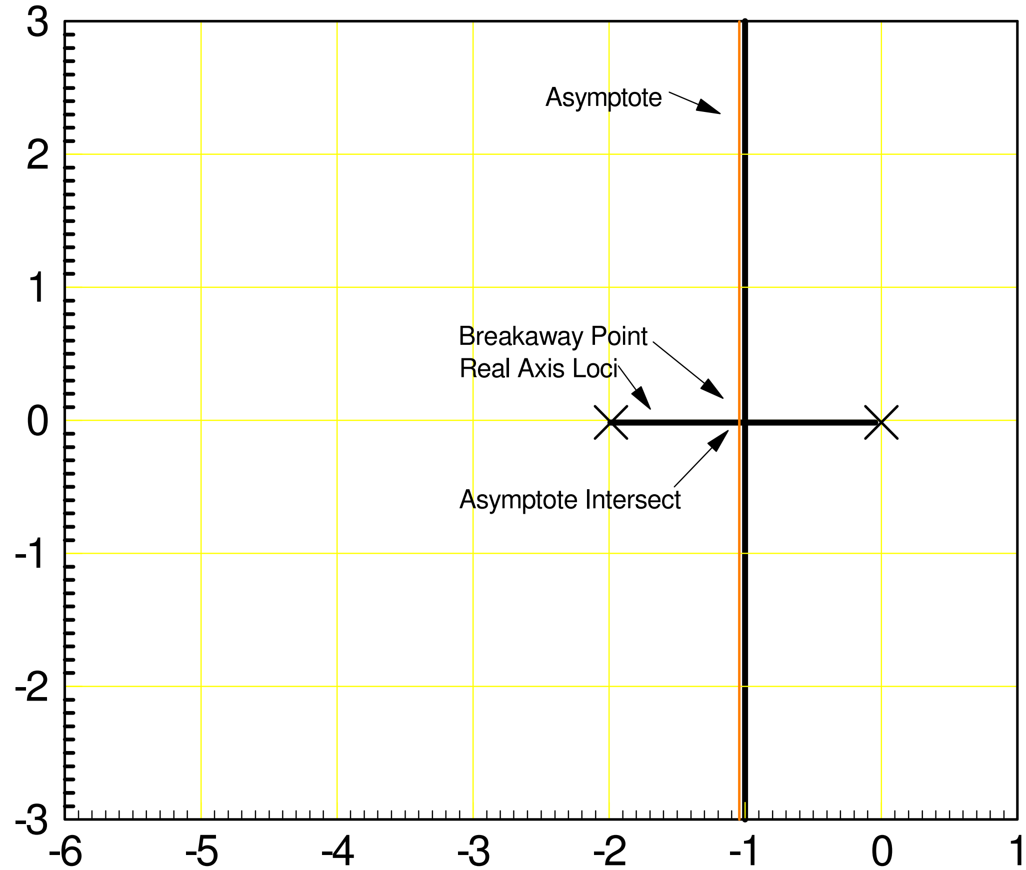
Gain at a point on the root-locus:

$$\left(\frac{kz(s)}{p(s)}\right) = kG(s) = -1 .$$

Root Locus Summary p(s) + k z(s) = 0	
Poles	solution to p(s)=0
Zeros	solution to z(s)=0
Real Axis Loci:	There are an off number of poles or zeros to the right
Breakaway Point(s)	$\frac{d}{ds}\left(\frac{p(s)}{z(s)}\right) = 0$
jw crossing	ang(G(jw)) = 180^0
Number of asymptotes	n-m (#poles - #zeros)
Asymptote Angles	$\frac{180^0 + k \cdot 360^0}{n-m}$
Asymptote Intersect	$\frac{\sum poles - \sum zeros}{n-m}$
Gain at any point on the root-locus	$k \cdot G(s) = -1$

Example 1: $G(s) = \left(\frac{1}{s(s+2)}\right)$

- Number of poles = 2 (n)
- Number of zeros = 0 (m)
- Open-Loop poles at {0, -2}
- Real Axis Loci: (0, -2)
- Number of asymptotes = 2
- Asymptote Angles = $\pm 90^0$
- Asymptote Intersect:
  - $\left(\frac{\sum poles - \sum zeros}{n - m}\right) = -1$
- Breakaway Point:
  - $\frac{d}{ds}(s(s+2)) = 2s + 2 = 0.$ s=-1.
- jw crossings: s=j0.



Describe how the previous system behaves as k increases from 0 to infinity:

For **k** small

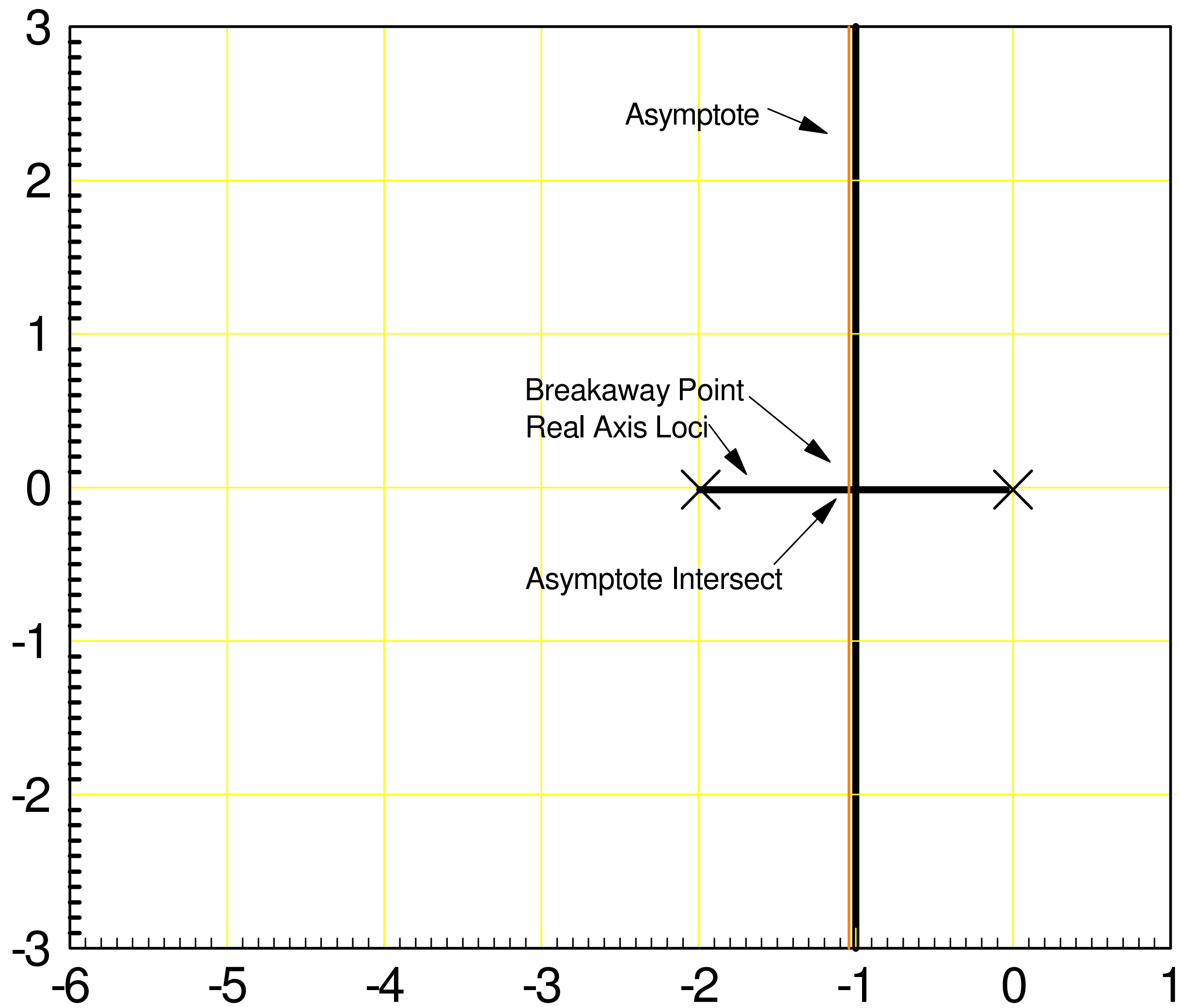
- slow first-order system.

k increases

- faster first order system
- limit is Ts = 4 sec (s = -1)

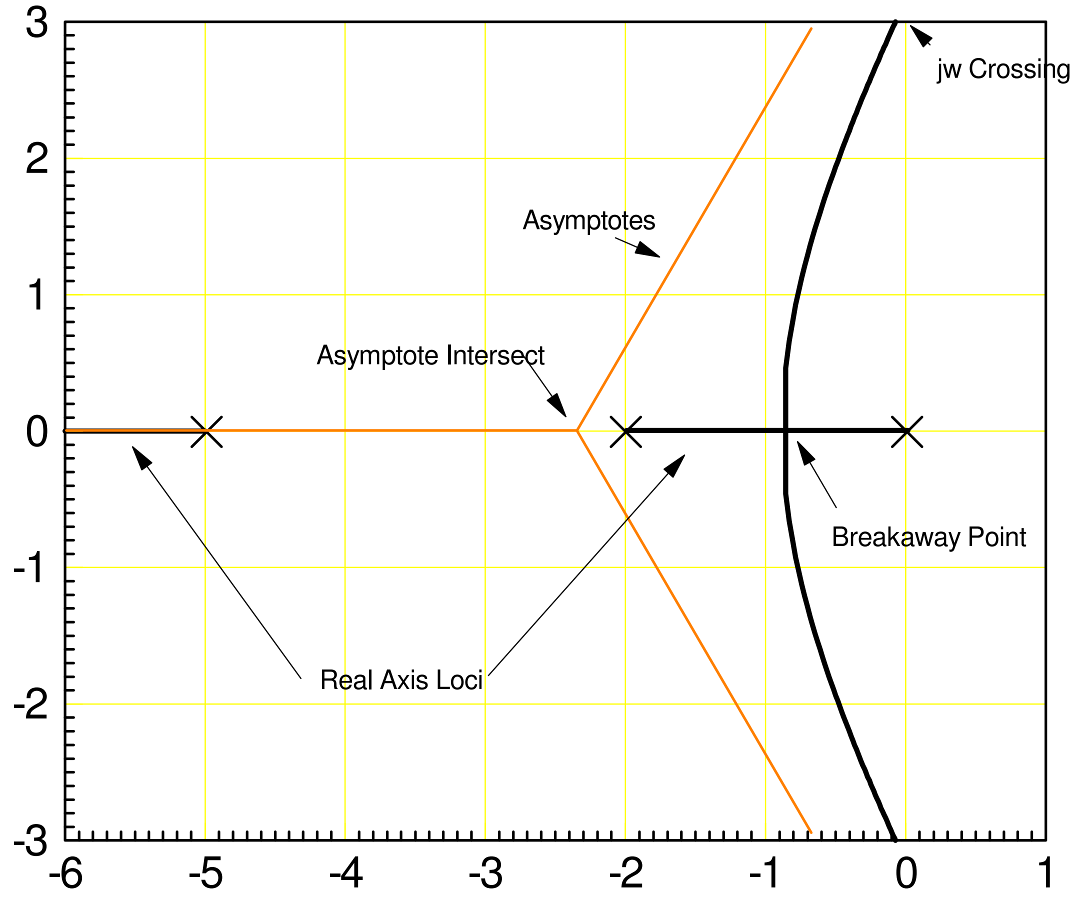
k increases more...

- start to get overshoot
- Ts = 4 seconds



Problem #2: $G(s) = \left(\frac{1}{s(s+2)(s+5)}\right)$

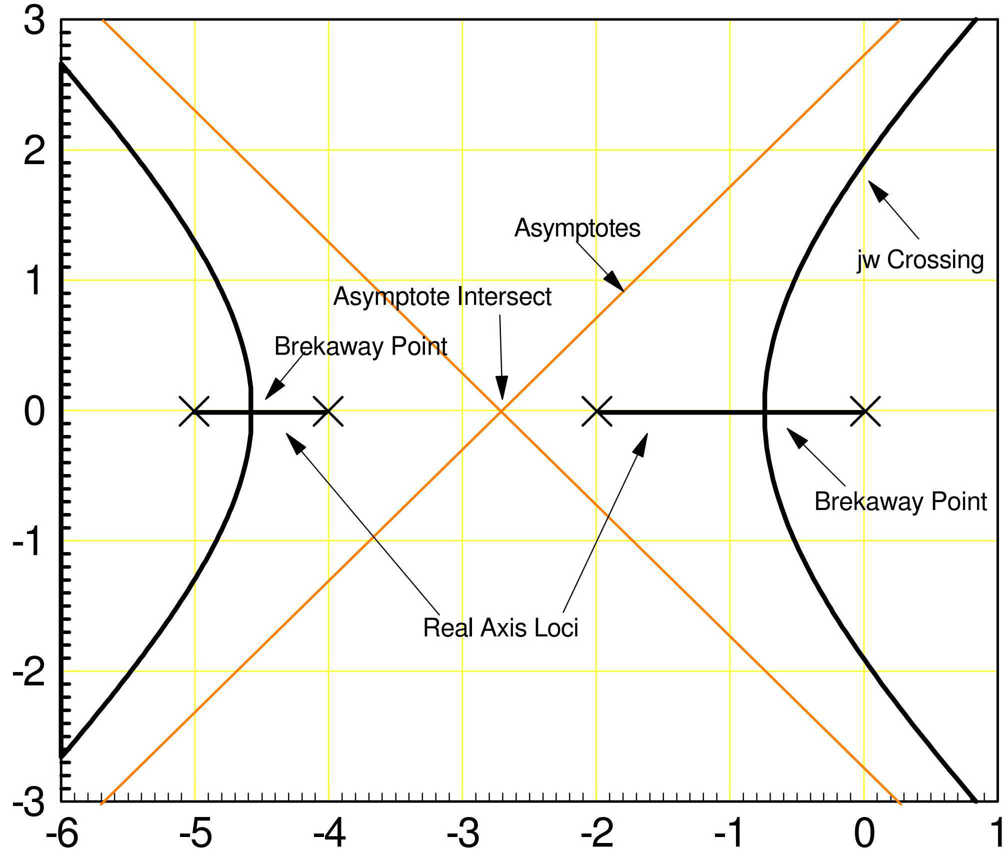
- 3 poles, 0 zeros
- Open-Loop poles at {0, -2, -5}.
- Real Axis Loci: (0, -2), (-5, -∞)
- Number of asymptotes = 3
- Asymptote Angles: $\pm 60^0, 180^0$
- Asymptote Intersect:
  - $\left(\frac{\sum poles - \sum zeros}{n-m}\right) = -2.3333$
- Breakaway Point:
  - $\frac{d}{ds}(s(s+2)(s+5)) = 0$
  - s = {-0.88, -3.79 }
- jw crossings: s=j3.16
  - $G(j3.16) = 0.0143\angle 180^0$



Example 3: $G(s) = \left(\frac{1}{s(s+2)(s+4)(s+5)}\right)$

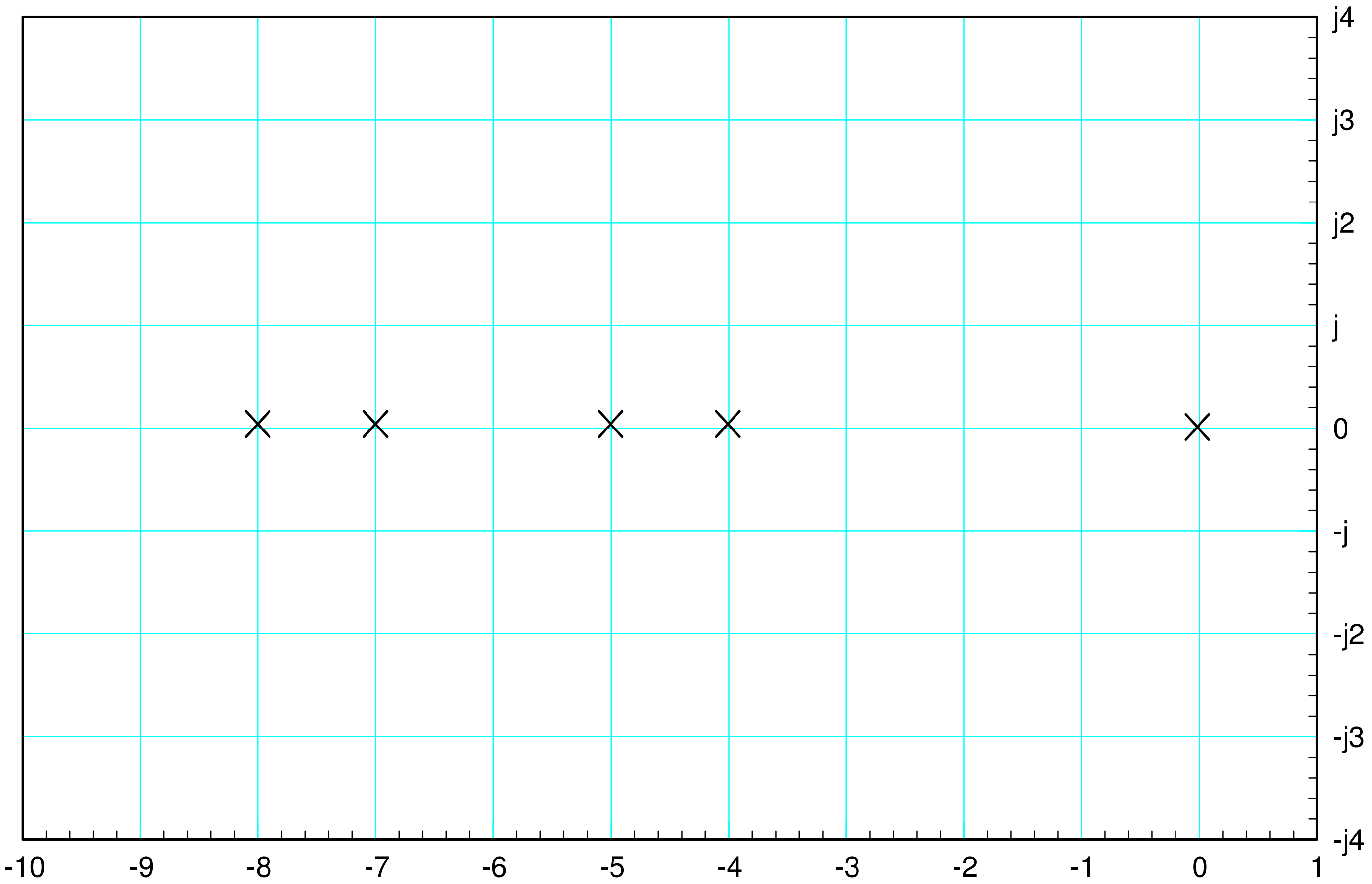
- 4 poles, no zeros
- Open-Loop poles at {0, -2, -4, -5}
- Open-Loop zeros: none
- Real Axis Loci: (0, -2), (-4, -5)
- Number of asymptotes = 4
- Asymptote Angles = $\pm 45^0, \pm 135^0$
- Asymptote Intersect:
  - $\frac{((0)+(-2)+(-4)+(-5))}{4} = -2.75$
- Breakaway Point:
  - $\frac{d}{ds}\left(\frac{s(s+2)(s+5)(s+10)}{s^2+2s+2}\right) = 0$
  - s = {-0.7438, -4.5771}
- jw crossings: $s = j1.9069$

$G(j1.9069) = 0.0080\angle 180^0$



Handout: Sketch the root locus for $G(s) = \left(\frac{2000}{s(s+4)(s+5)(s+7)(s+8)}\right)$

Real Axis Loci:



Breakaway Point(s)

jw Crossing

Asymptotes

Summary

If you adjust the gain, k, for a unity feedback system, the closed-loop poles shift

These shifting poles follow a well-defined path

- Termed the root-locus plot

By sketching the root locus, you know what options are available for the closed-loop system