Meeting Design Specs Using Root Locus

ECE 461/661 Controls Systems

Jake Glower - Lecture #26

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Lead & PID Compensators

Lead

- Pull the root locus left, speeding up the system

PID

- Add a pole at s = 0 (making Type-0 systems Type-1)
- Add 0, 1, or 2 zeros

In General

- Add a pole at s = 0 (if needed) to make the system type-1
- Add zeros to cancel poles, speeding up the system, and
- Add fast poles so that #poles >= # zeros

The resulting K(s) may not have name per say.

Example: $G(s) = \left(\dfrac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)} \right)$

Find K(s) so that there is...

- No error for a step input,
- A 2% settling time of 4 seconds,
- 20% overshoot for a step input, and
- The high-frequency gain of K(s) is finite

Translating:

- GK is type-1 (or more)
- Closed-loop dominant pole: $s = -1 + j2$
- #poles ≥ #zeros

$$G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)} \right)$$

Step 1) Add a pole at s = 0

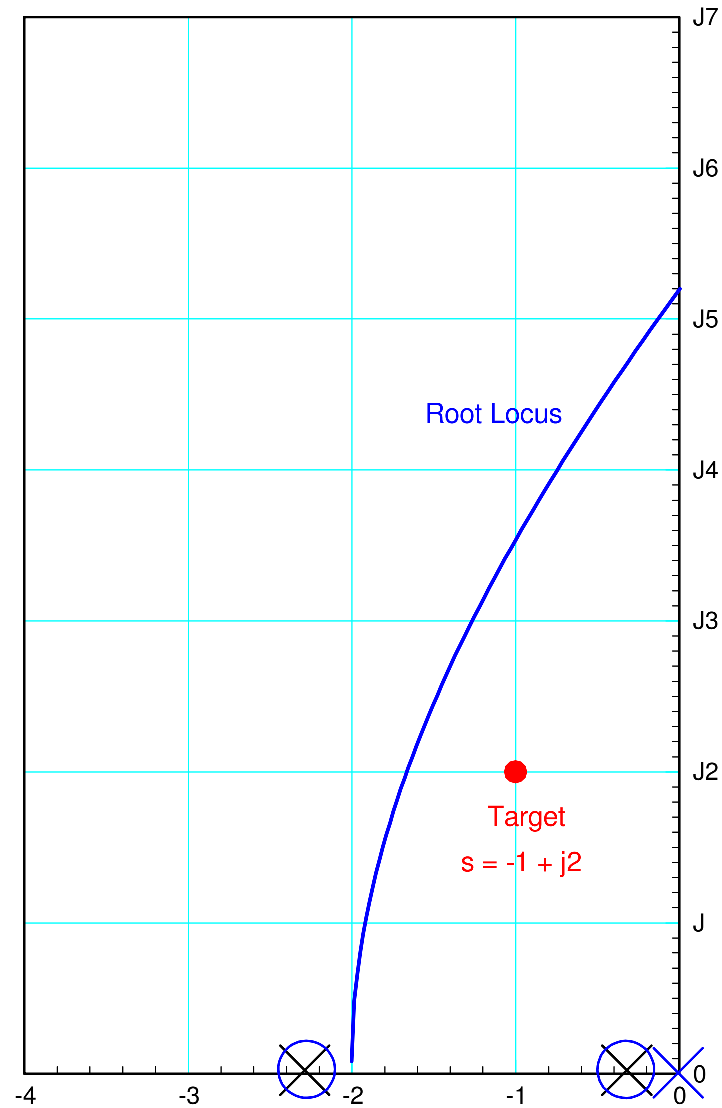
- Makes GK type-1

$$K(s) = \left(\frac{k}{s} \right)$$

Step 2) Start canceling poles

- Keep going until you're too fast
- The root locus passes left of design point

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s} \right)$$



Step 3) Add a pole (#poles = #zeros)

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s(s+a)}\right)$$

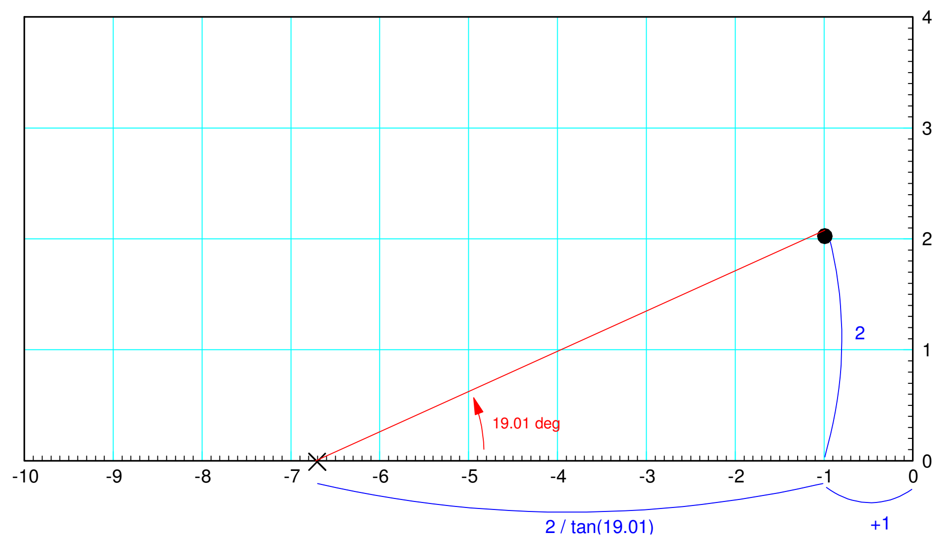
Pick 'a' so that s = -1 + j2 is on the root locus

$$GK = \left(\frac{361.2378k}{s(s+5.439)(s+10.1)(s+15.65)(s+a)}\right)_{s=-1+j2} = 1\angle 180^0$$

$$\left(\frac{361.2378}{s(s+5.439)(s+10.1)(s+15.65)}\right)_{s=-1+j2} = 0.2409\angle -160.99^0$$

$$\angle(s+a) = 19.01^0$$

$$a = \frac{2}{\tan(19.01^0)} + 1 = 6.8046$$



Step 4: Find k

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s(s+6.8046)}\right)$$

$$GK = \left(\frac{361.2378}{s(s+5.439)(s+6.8046)(s+10.1)(s+15.65)}\right)_{s=-1+j2}$$

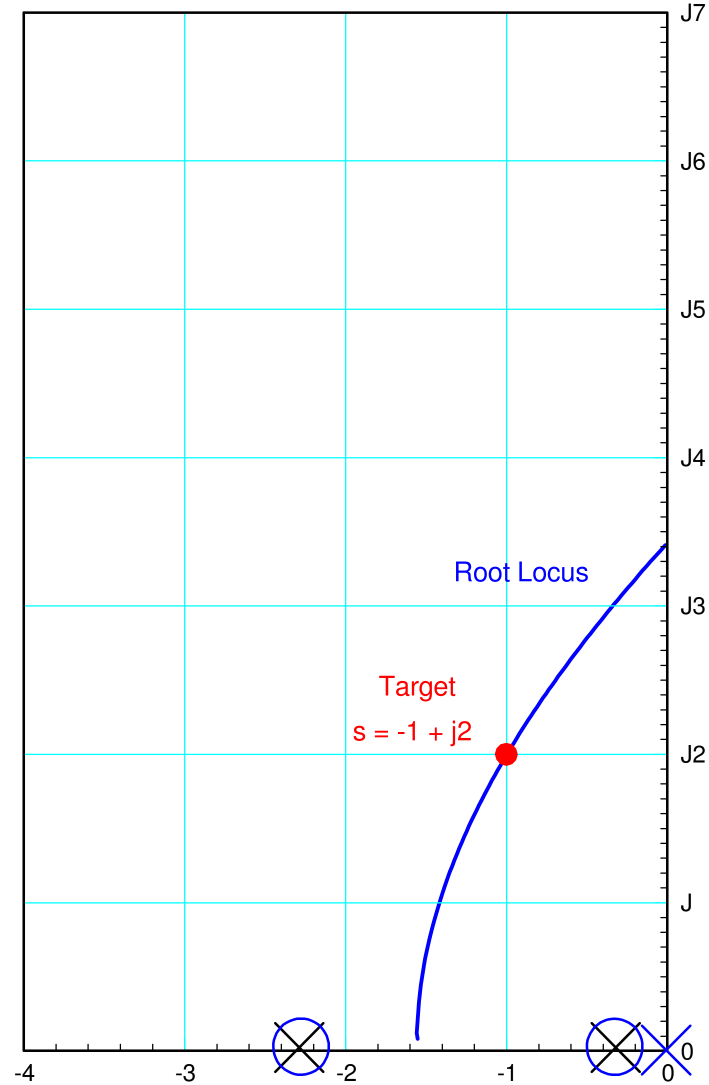
$$GK = 0.0392\angle 180^0$$

- 180 degrees: -1 + j2 is on the root locus (good)
- 0.0393: k is off (gain should be 1.000)

$$k = \frac{1}{0.0392} = 25.49$$

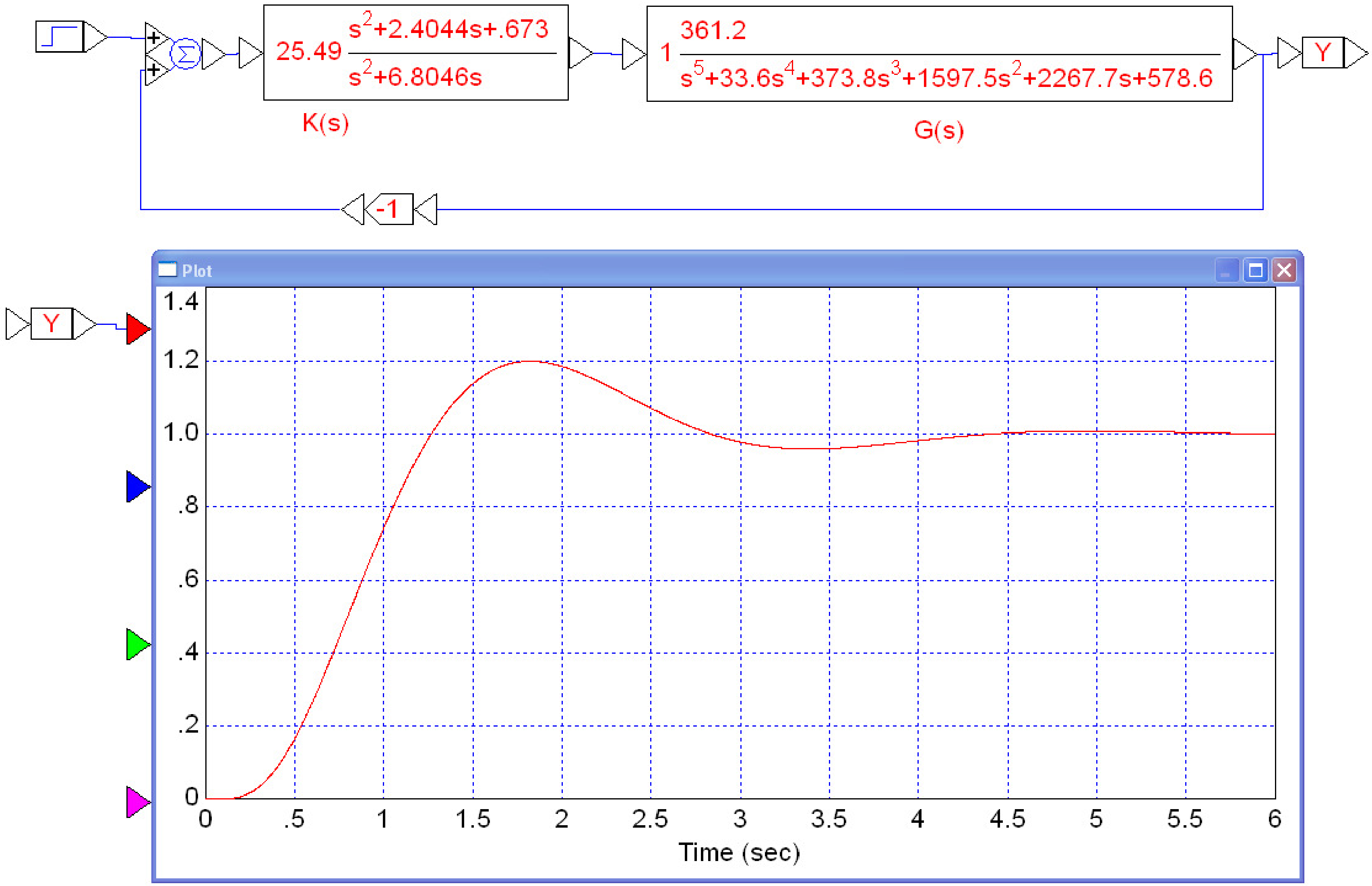
so

$$K(s) = 25.49\left(\frac{(s+0.3234)(s+2.081)}{s(s+6.8046)}\right)$$



Check in Matlab:

```
 G = zpk([],[-0.3234, -2.081, -5.439, -10.1, -15.65], 361.2378 )

                    361.2378
-------------------------------------------------
(s+0.3234) (s+2.081) (s+5.439) (s+10.1) (s+15.65)

 K = zpk([-0.3234, -2.081],[0, -6.8046], 25.49 )

25.49 (s+0.3234) (s+2.081)
--------------------------
       s (s+6.805)

 Gcl = minreal( G*K / (1 + G*K) )
 eig(Gcl)

  -1.0000 + 2.0000i             pole we placed
  -1.0000 - 2.0000i
  -9.7606 + 4.0657i
  -9.7606 - 4.0657i
 -16.4724
```

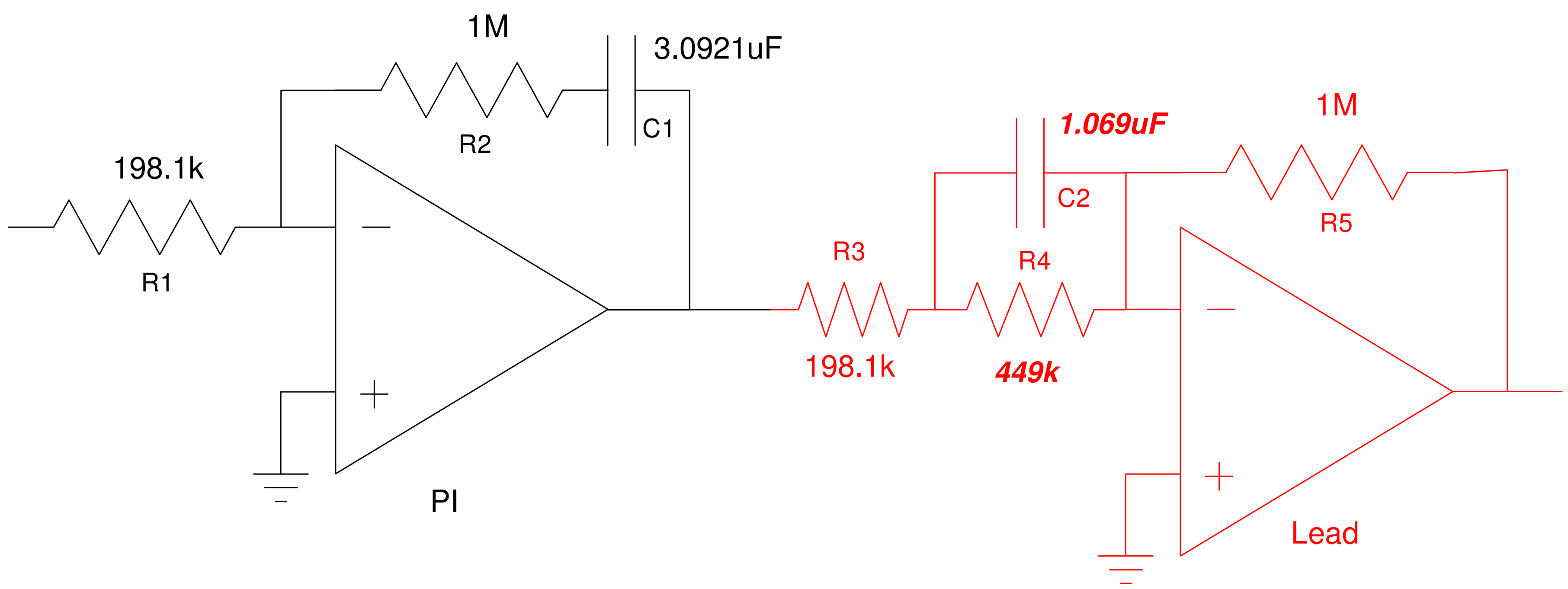


25.49
s^2+2.4044s+.673
s^2+6.8046s
K(s)
1
361.2
s^5+33.6s^4+373.8s^3+1597.5s^2+2267.7s+578.6
G(s)
Y
-1
Plot
1.4
1.2
1.0
.8
.6
.4
.2
0
0
.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Time (sec)

Circuit Implementation:

Rewrite as

$$K(s) = \left(5.05\frac{s+0.3234}{s}\right)\left(5.05\frac{s+2.081}{s+6.8046}\right)$$

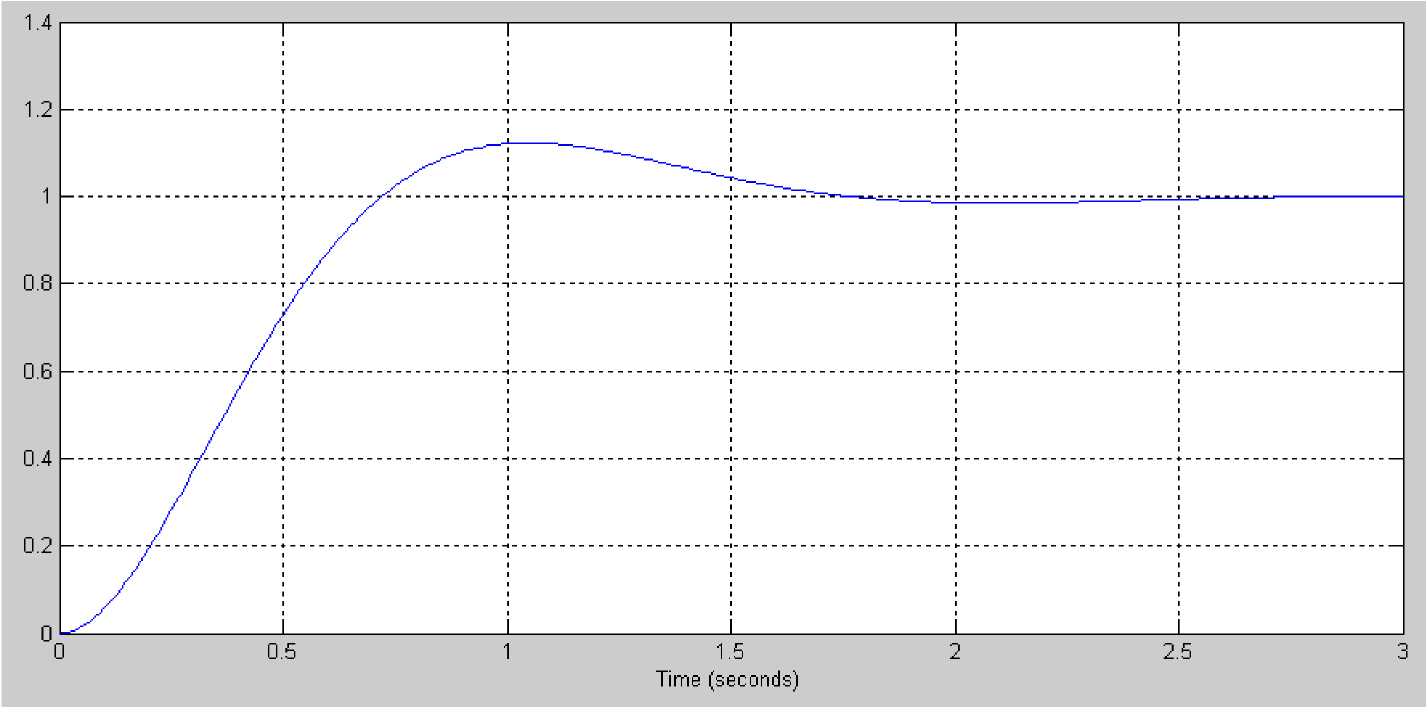


Example 2:

Let

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+5)(s+10)}\right)$$

Design a compensator so that the closed-loop step response looks like this:

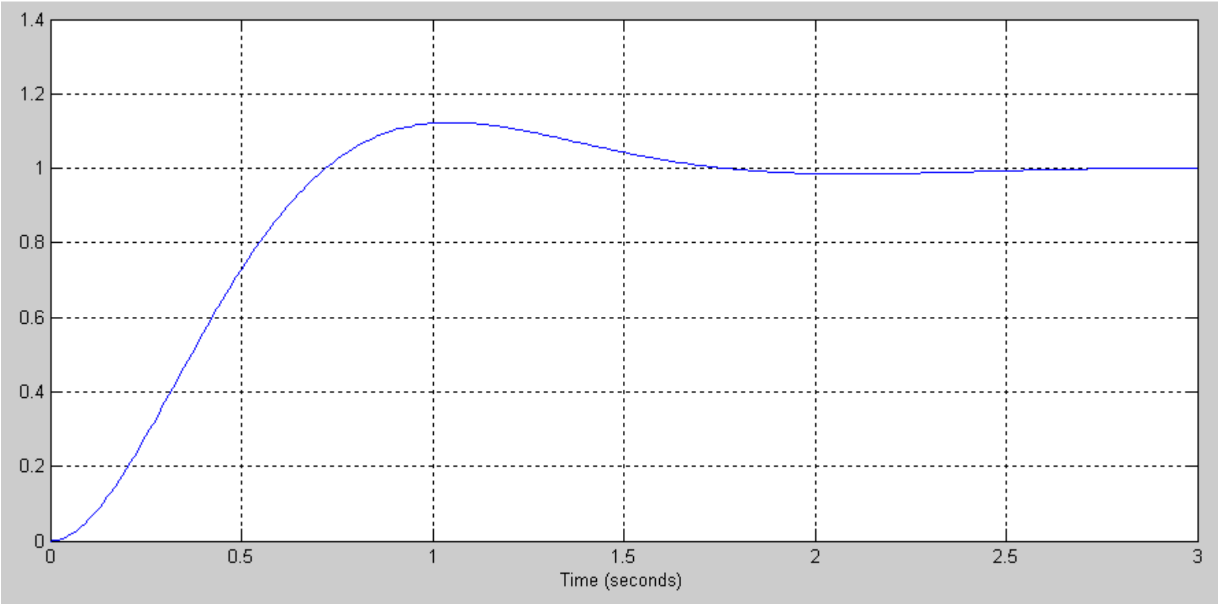


Desired Step Response

Solution:

Step 1: Translate the requirements to root-locus terms.

- The DC gain is one: the system should be type-1
- The 2% settling time is 2 seconds:
  - the real part of the closed-loop dominant pole should be -2
- The overshoot is 12%
  - The damping ratio should be $\zeta = 0.55$
  - The closed-loop dominant pole should be s = -2 + j3



Step 2: Find K(s)

- Make the system type-1
- s = -2 + j3 is on the root locus

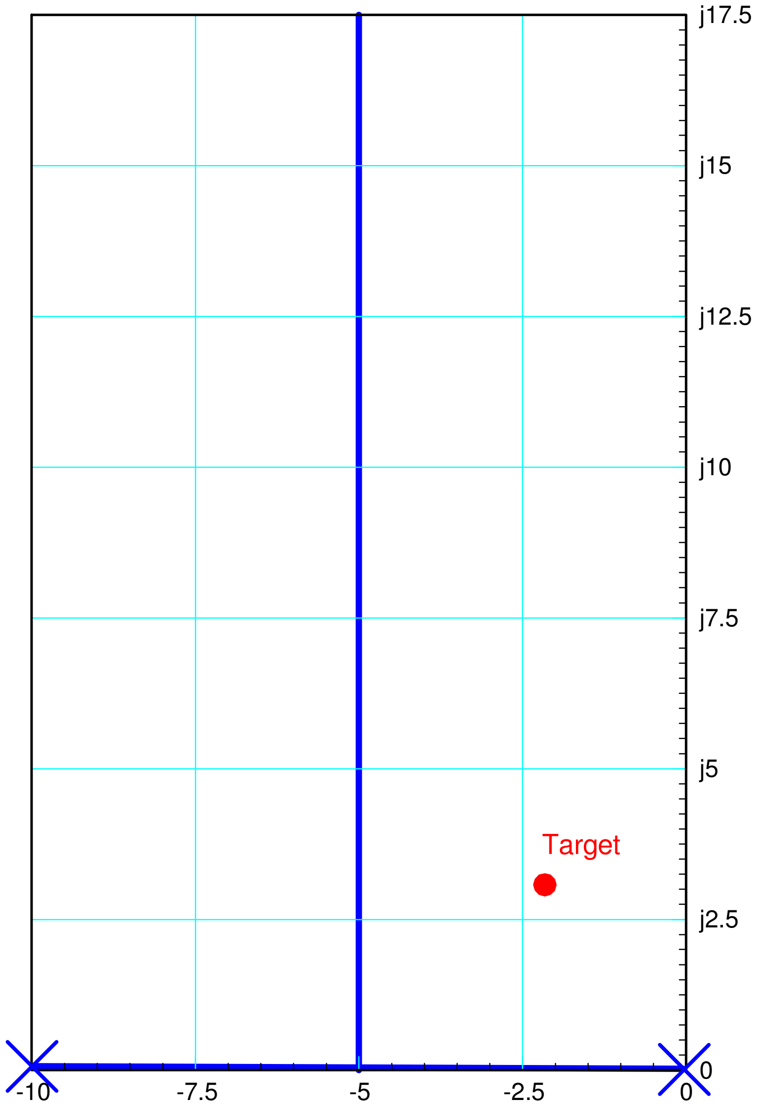
Start with

$$K(s) = \frac{1}{s}$$

Start canceling zeros until you're too fast

$$K(s) = \left(\frac{(s+1)(s+3)(s+5)}{s(s+a)^2}\right)$$

$$GK = \left(\frac{100}{s(s+10)}\right)$$



Add two poles so that #poles = #zeros

$$K(s) = \left(\frac{(s+1)(s+3)(s+5)}{s(s+a)^2} \right)$$

$$GK = \left(\frac{100}{s(s+a)^2(s+10)} \right)$$

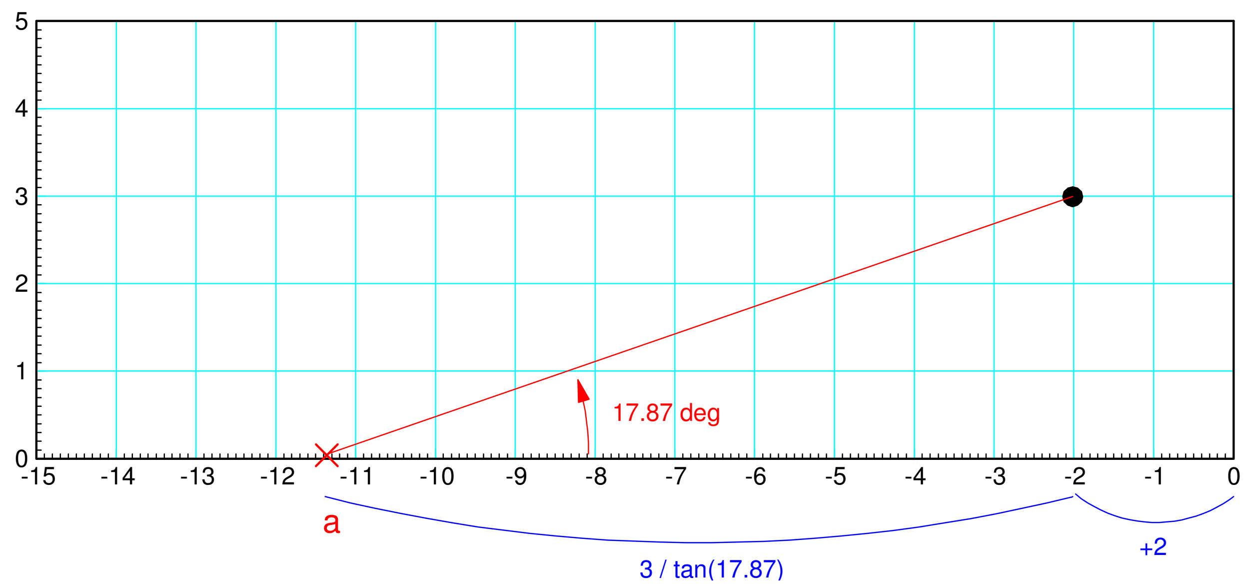
Find 'a' so that the angles add up

$$\left(\frac{100}{s(s+a)^2(s+10)} \right)_{s=-2+j3} = X\angle 180^0$$

$$\left(\frac{100}{s(s+10)} \right)_{s=-2+j3} = 3.2461\angle -144.26^0$$

$$\angle(s+a) = 17.87^0$$

$$a = \frac{3}{\tan(17.87^0)} + 2 = 11.301$$



This results in

$$K(s) = \left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2}\right)$$

$$GK = \left(\frac{100k}{s(s+11.301)^2(s+10)}\right)$$

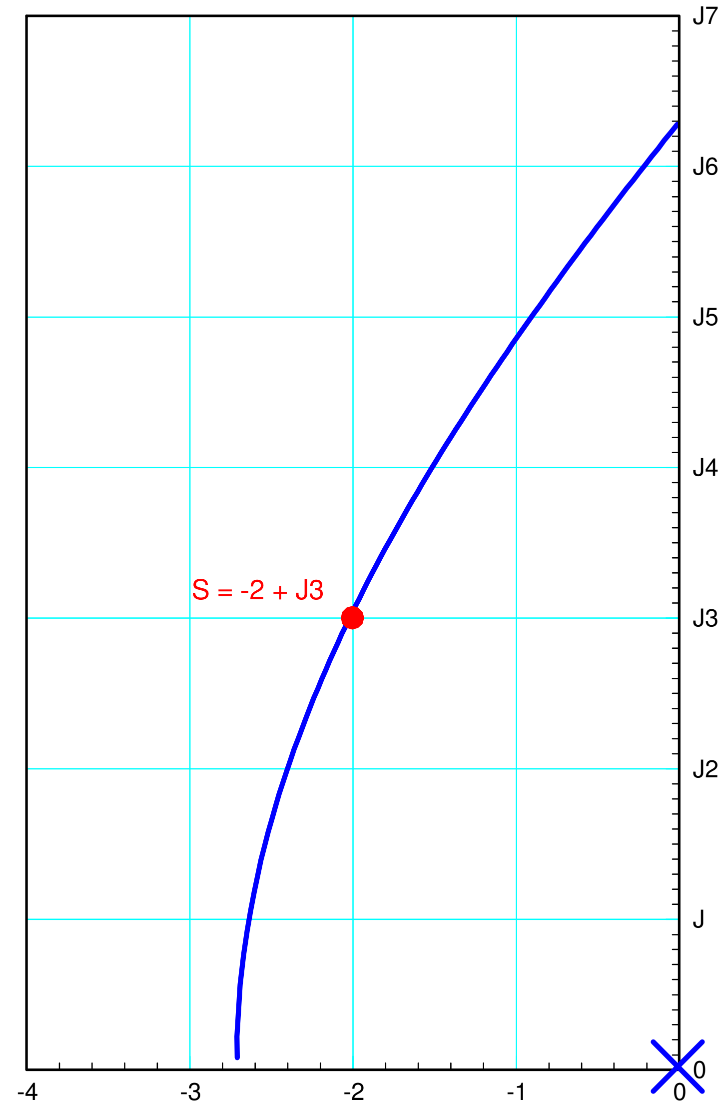
Step 3: Find k

$$\left(\frac{100}{s(s+11.301)^2(s+10)}\right)_{s=-2+j3} = 0.0340\angle 180^0$$

$$k = \frac{1}{0.0340} = 29.4222$$

and

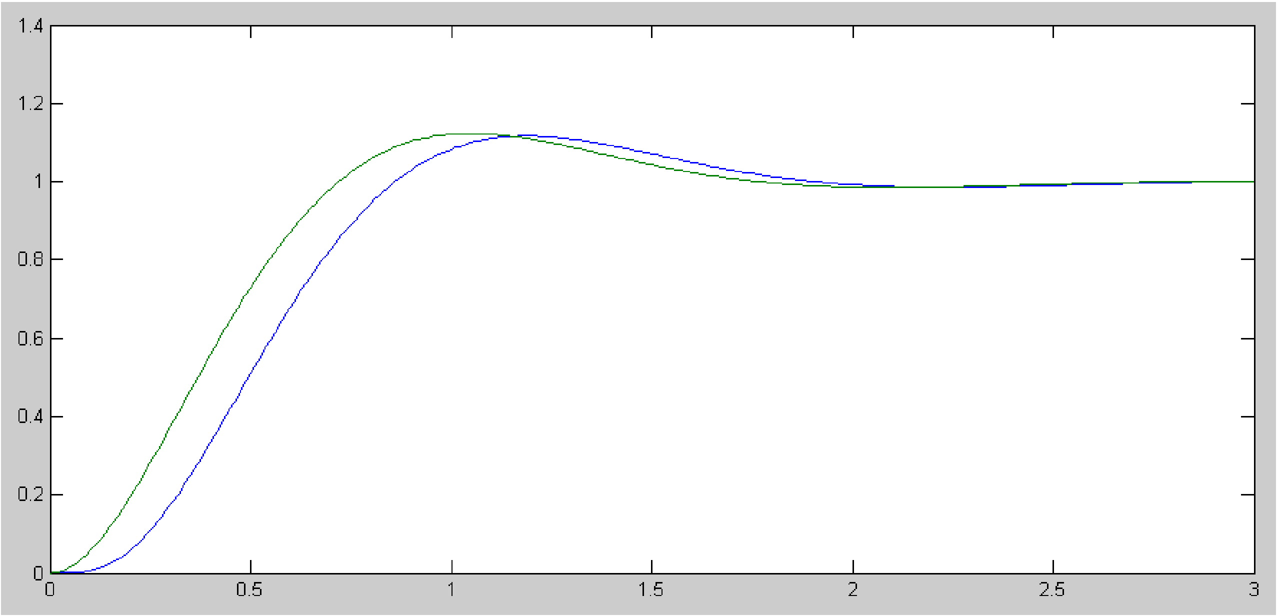
$$\boxed{K(s) = 29.4222\left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2}\right)}$$



Validation in Matlab

- Actual vs. Desired Step Response

```
Gd = zpk([],[-2+j*3,-2-j*3],13);

G = zpk([],[-1,-3,-5,-10],100);
K = zpk([-1,-3,-5],[0,-11.301,-11.301],29.4222);
Gcl = minreal(G*K / (1+G*K));

eig(Gcl)

-14.3010 + 4.6697i
-14.3010 - 4.6697i
 -2.0000 + 3.0000i
 -2.0000 - 3.0000i

t = [0:0.01:3]';
yd = step(Gd, t);
y = step(Gcl,t);
plot(t,y,t,yd)
```



Circuit:

$$K(s) = 29.4222\left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2}\right)$$

Split into three terms:

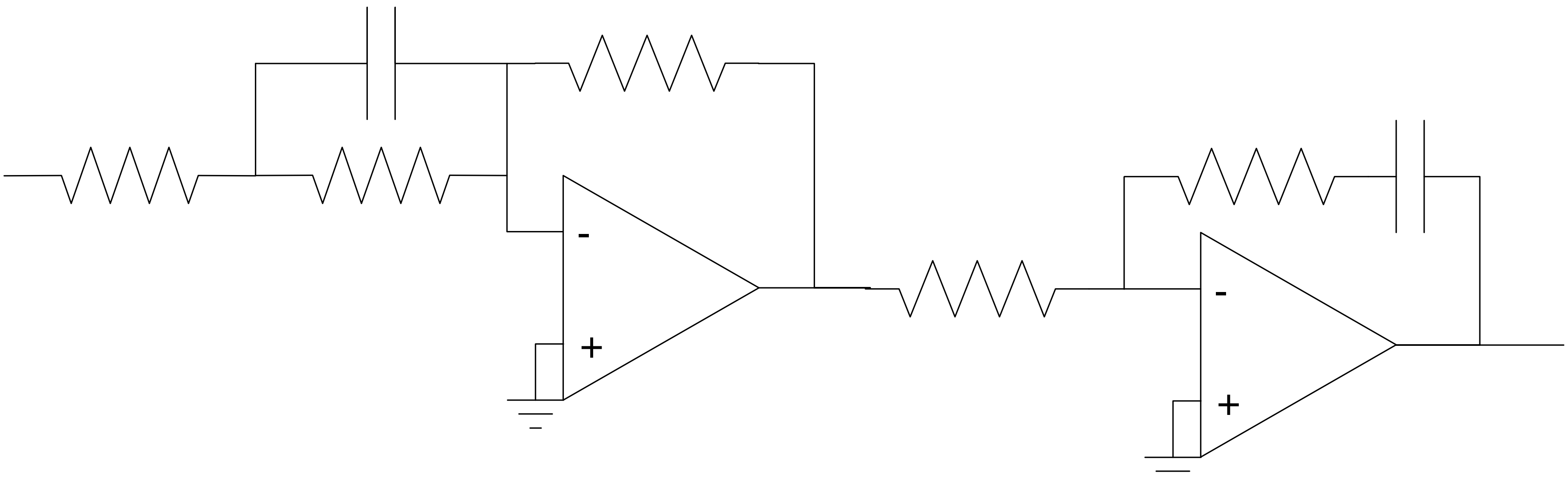
$$K(s) = \left(3.087\left(\frac{s+1}{s}\right)\right)\left(3.087\left(\frac{s+3}{s+11.301}\right)\right)\left(3.087\left(\frac{s+5}{s+11.301}\right)\right)$$

This would then be a PI * Lead * Lead compensator

Handout

Find R and C to implement K(s)

$$K(s) = \left(\frac{20(s+2)(s+5)}{s(s+10)}\right)$$



Summary

To design a compensator to meet your design specs...

- Translate the specs to root-locus terms
  - Type-1 = no error for a step input
  - real(s) >> 2% settling time
  - angle(s) >> overshoot
- Add a pole at s = 0 if it's a type-0 system
  - Don't add a second pole at s = 0
- Start cancelling (stable) zeros until you're too fast
- Then add poles to force the root locus through your design point
  - #poles = #zeros for K(s)