Converting G(s) to G(z)

ECE 461/661 Controls Systems

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Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Digital Control Systems

Often times, the compensator, K(s), will be implemented with a microcontroller.

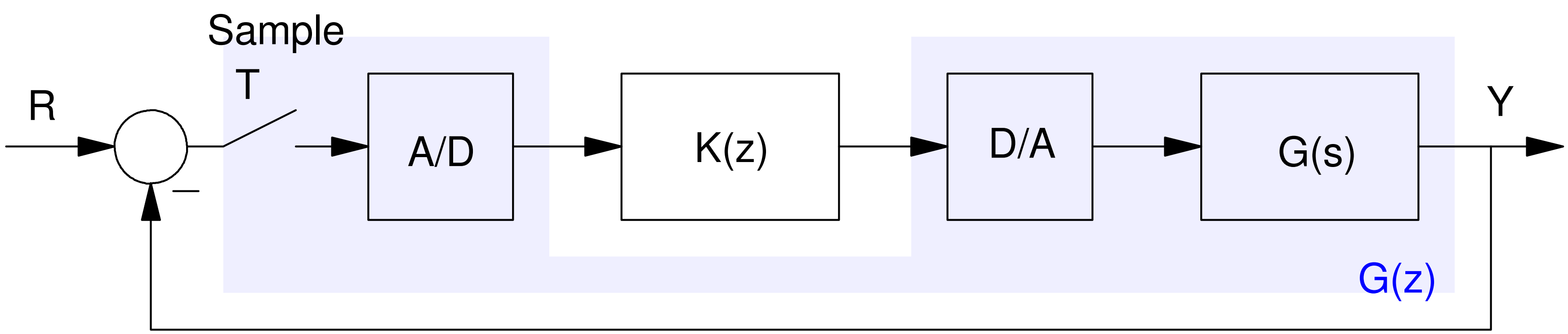
- The result is a hybrid system: both analog and discrete

  Makes analysis difficult

Relative to the microcontroller, the world looks discrete:

- You output a control signal every T seconds,
- Through the system dynamics, this results in an error, sampled every T seconds as well.
- Convert everything to the z-domain

  Now you can analyze the closed-loop system



Converting G(s) to G(z)

Method 1: Substitution

- sY means *the derivative of Y*

Euler - Backward Difference

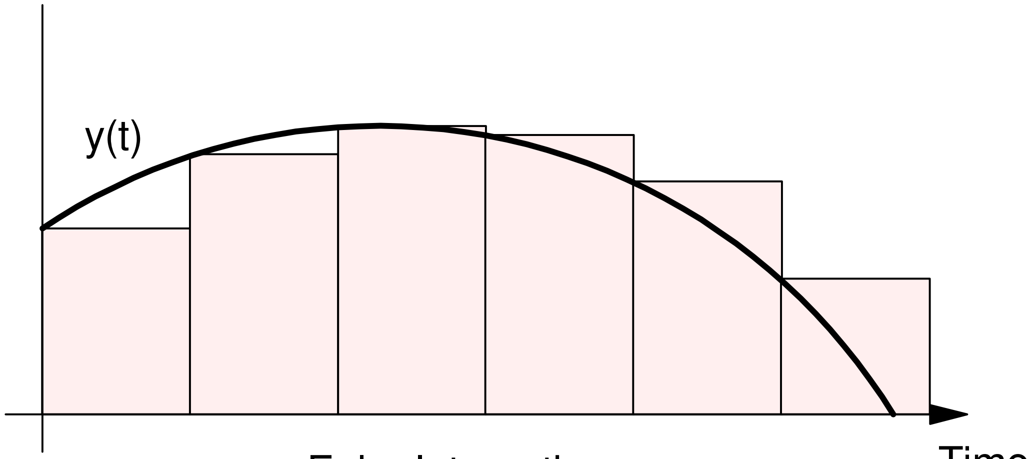
$$s \approx \left(\frac{z-1}{T}\right)$$

Euler - Forward Difference:

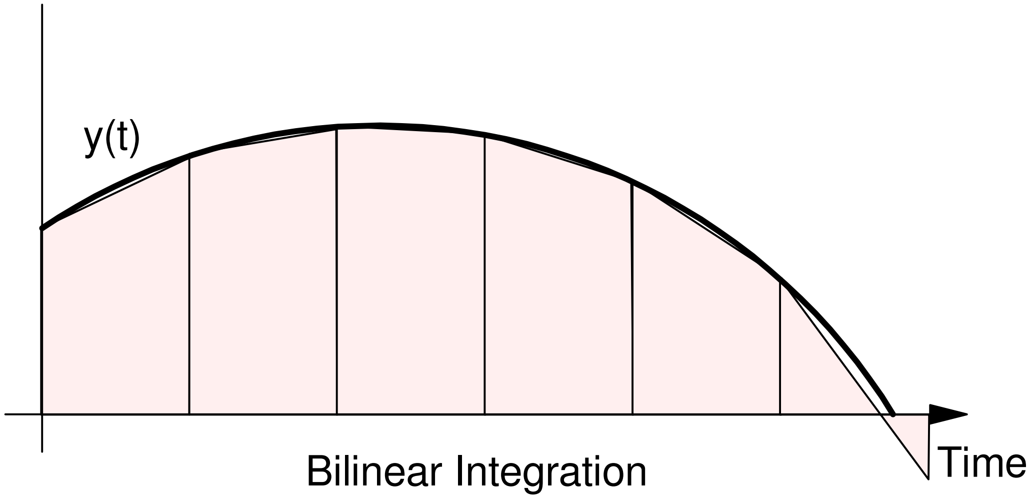
$$s \approx \left(\frac{z-1}{Tz}\right)$$

Bilinear

$$s \approx \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$



Euler Integration

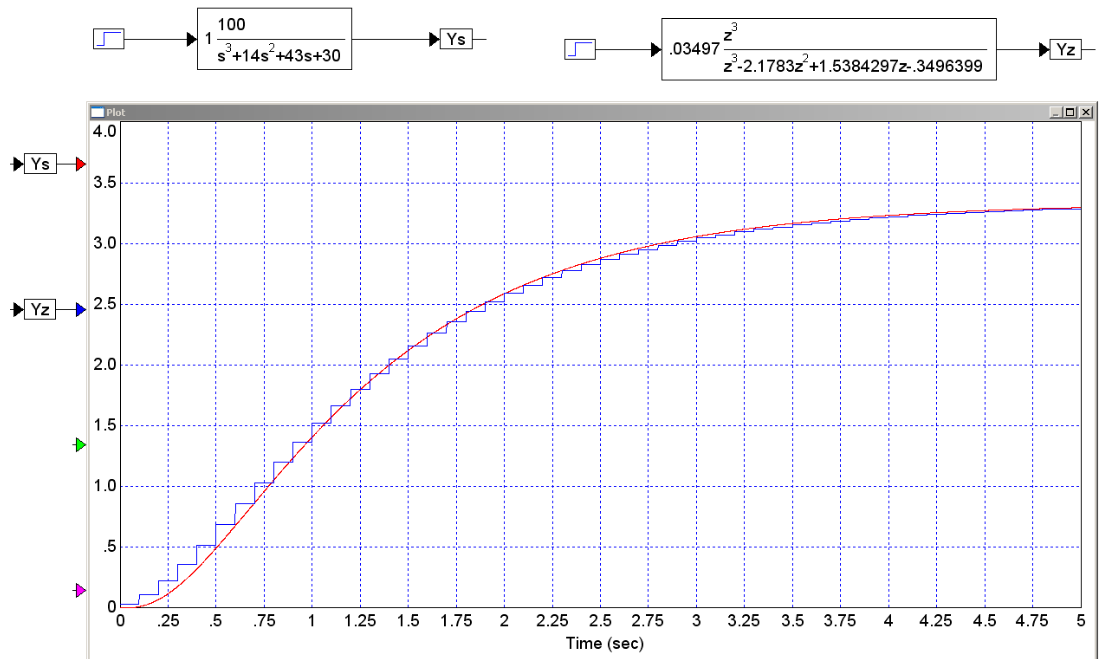


Bilinear Integration

Example: Find the z-transform of G(s)

- Assume a sampling rate of 100ms (T = 0.1).

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$



a) Using Euler Forward Difference:

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

$$G(z) \approx \left(\frac{100}{\left(\left(\frac{z-1}{Tz}\right)+1\right)\left(\left(\frac{z-1}{Tz}\right)+3\right)\left(\left(\frac{z-1}{Tz}\right)+10\right)} \right)$$

$$G(z) = \left(\frac{0.03497z^3}{(z-0.9091)(z-0.7692)(z-0.5)} \right)$$

In Matlab: Input the system in the s-plane

```
Gs = zpk([],[-1,-3,-10],100)

       100
------------------
(s+1) (s+3) (s+10)
```

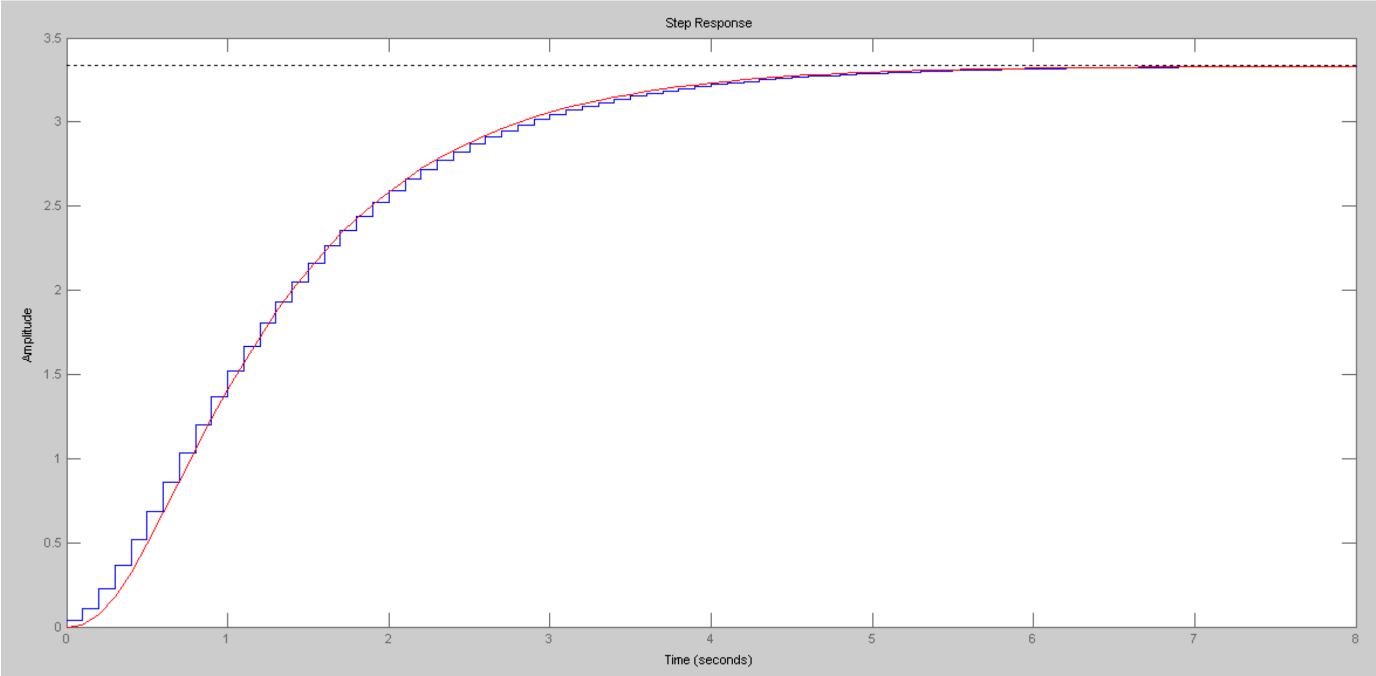
If you add one more term, Matlab interprets this as a discrete-time system with the last term being the sampling rate:

```
Gz = zpk([0,0,0],[0.9091,0.7692,0.5],0.03497,0.1)

         0.03497 z^3
-----------------------------
(z-0.9091) (z-0.7692) (z-0.5)

Sampling time (seconds): 0.1
```

To plot the step response of the two systems together

- Plot the step response of G(z)
- Type *hold on* to keep this plot
- Plot the step response of G(s) on top of the G(z) graph

```
step(Gz)

t = [0:0.01:8]';
ys = step(Gs,t);
hold on
plot(t,ys,'r')
```

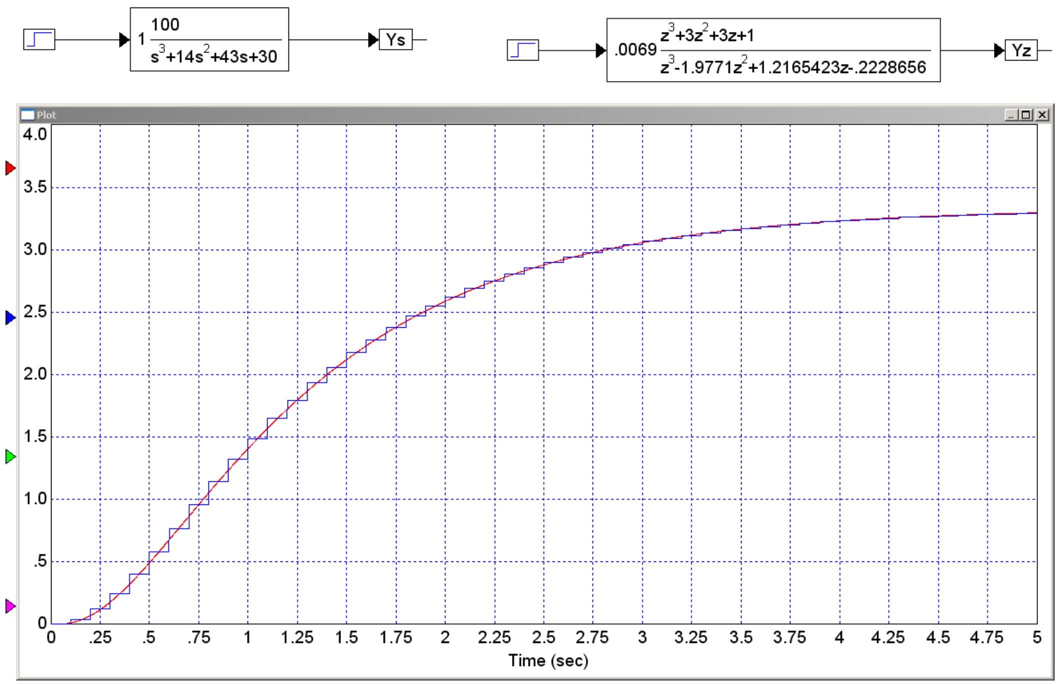


Bilinear:

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)} \right)$$

$$G(z) \approx \left(\frac{100}{\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+1\right)\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+3\right)\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+10\right)} \right)$$

$$G(z) \approx \left(\frac{0.00690(z+1)^3}{(z-0.9047)(z-0.7391)(z-0.3333)} \right)$$



Method #2: Transform Poles and Zeros (my preference)

LaPlace transforms assume

$$y = e^{st}$$

z-transforms assume

$$y = z^k$$

Assume

$$t = kT$$

$$y = e^{s(kT)} = e^{(sT)k} = \left(e^{sT}\right)^k = z^k$$

The conversion from the s-plane to the z-plane is

$$z = e^{sT}$$

Procedure:

i) Convert every pole and zero as

$$z = e^{sT}$$

ii) Add a gain to match the DC gain

iii) (optional) Add n zeros at z = 0 to match the phase at a frequency close to zero

- or -

Add n zeros at z = 0 to match the delay in the system.

Example: G(z). Assume T = 0.1

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)}\right)$$

In Matlab

```
s = [-1, -3, -10]

   -1.00     -3.00     -10.00

T = 0.1;

z = exp(s*T)

    0.9048374      0.7408182      0.3678794
```

Meaning

$$G(z) = \left(\frac{k}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)$$

Match the DC gain

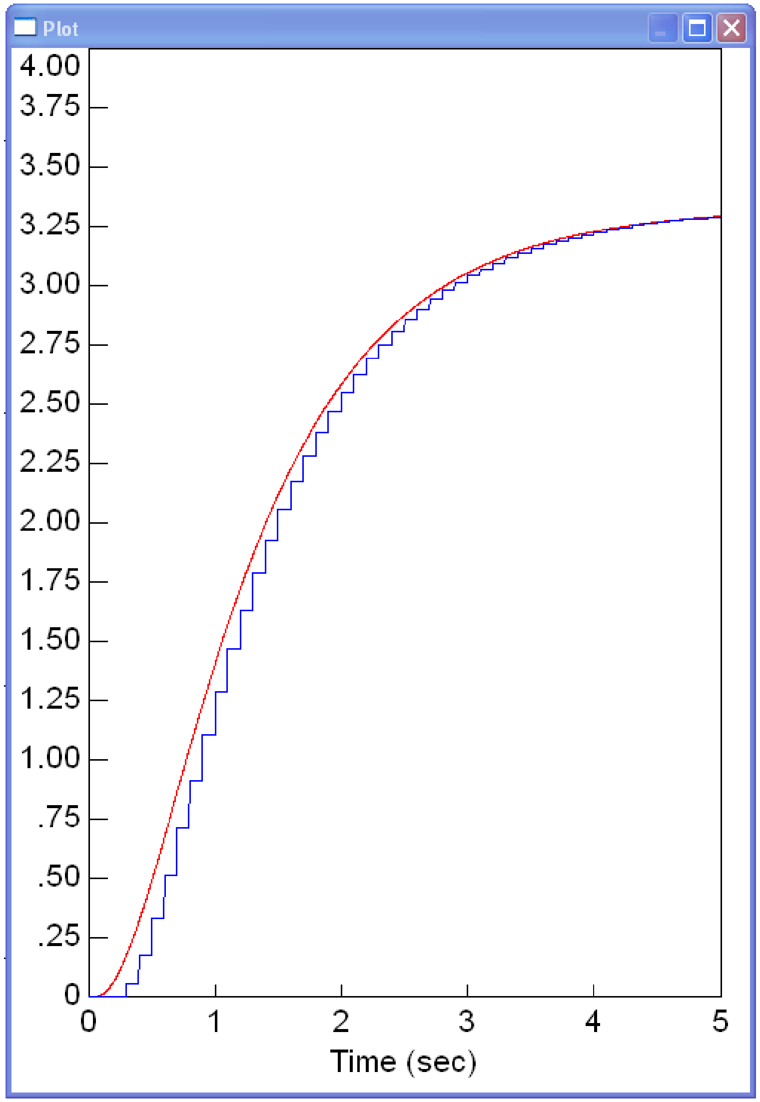
$$\left(\frac{100}{(s+1)(s+3)(s+10)}\right)_{s=0} = 3.3333$$

$$\left(\frac{k}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)_{z=1} = 3.33333$$

```
k = prod(1-z) * 3.3333

    0.0591691
```

meaning

$$G(z) = \left(\frac{0.059169}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)$$



To find how many zeros belong at z=0,

a) There is too much delay with this system. Adjust by adding zeros at z = 0

b) Match the phase at some frequency, such as s = j1

$$\left(\frac{100}{(s+1)(s+3)(s+10)}\right)_{s=j1} = 2.2249\angle -69.14^0$$

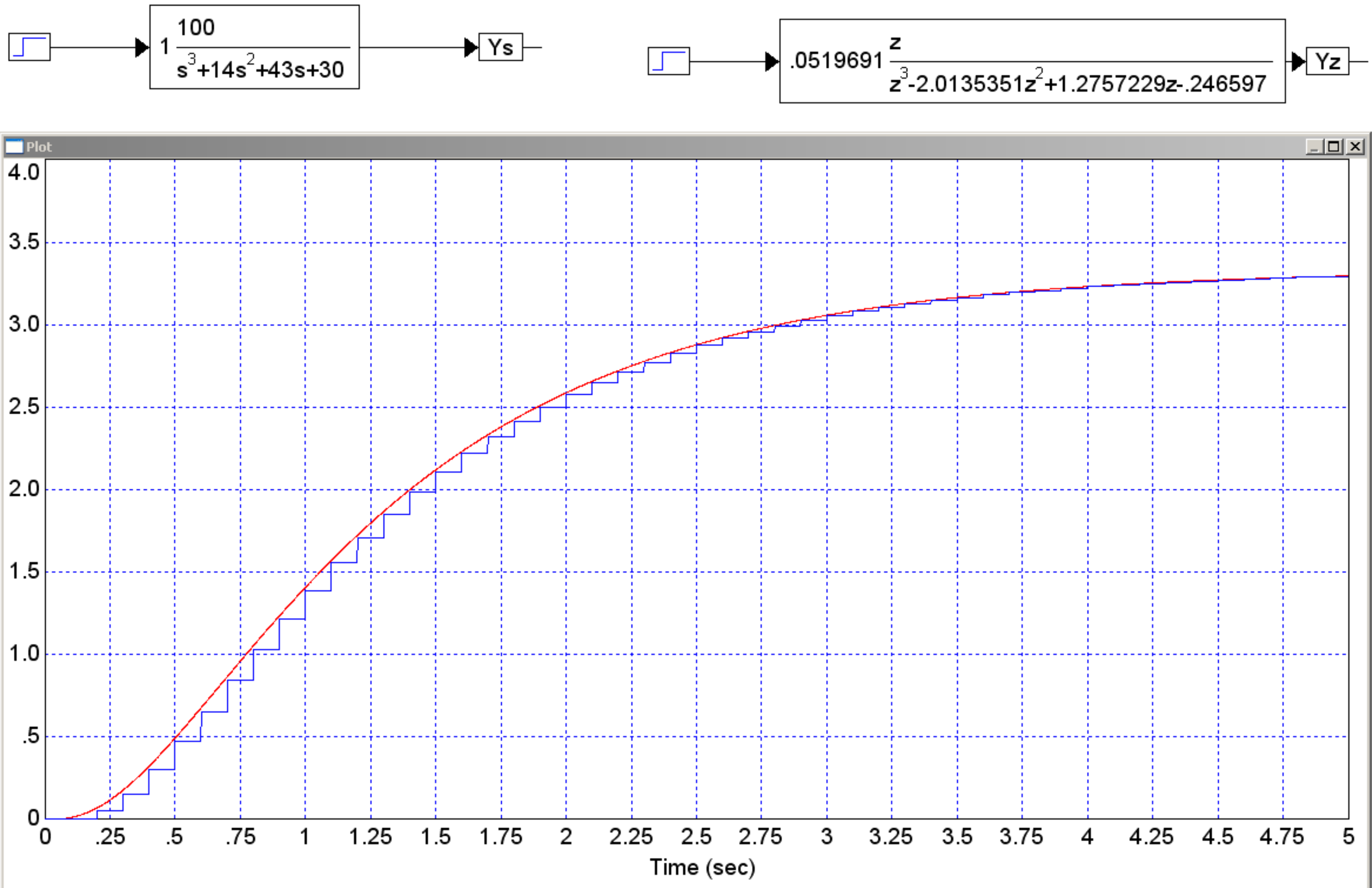
$$\left(\frac{0.051969}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)_{s=j} = 2.2276\angle -78.04^0$$

$$z = e^{sT} = e^{(j1)(0.1)} = 1\angle 5.73^0$$

Add 1.55 zeros at z=0 to make the phase match

- Round up to 2 or round down to 1

$$G(z) \approx \left(\frac{0.051969\cdot z}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)$$



1 100 / s^3+14s^2+43s+30
Ys
.0519691 z / z^3-2.0135351z^2+1.2757229z-.246597
Yz
Plot
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
0
.25
.5
.75
1
1.25
1.5
1.75
2
2.25
2.5
2.75
3
3.25
3.5
3.75
4
4.25
4.5
4.75
5
Time (sec)

In Matlab: Input the system G(s)

```
Gs = zpk([],[-1,-3,-10],100)

       100
------------------
(s+1) (s+3) (s+10)
```

Input G(z). For now, assume the numerator is 1

```
T = 0.1;
Gz = zpk([],[exp(-1*T),exp(-3*T),exp(-10*T)],1,T)

               1
--------------------------------
(z-0.9048) (z-0.7408) (z-0.3679)

Sampling time (seconds): 0.1
```

Add a gain, k, so that the DC gain matches up

```
DCs = evalfr(Gs,0)

    3.3333

DCz = evalfr(Gz,1)

   64.1401

k = DCs / DCz

    0.0520
```

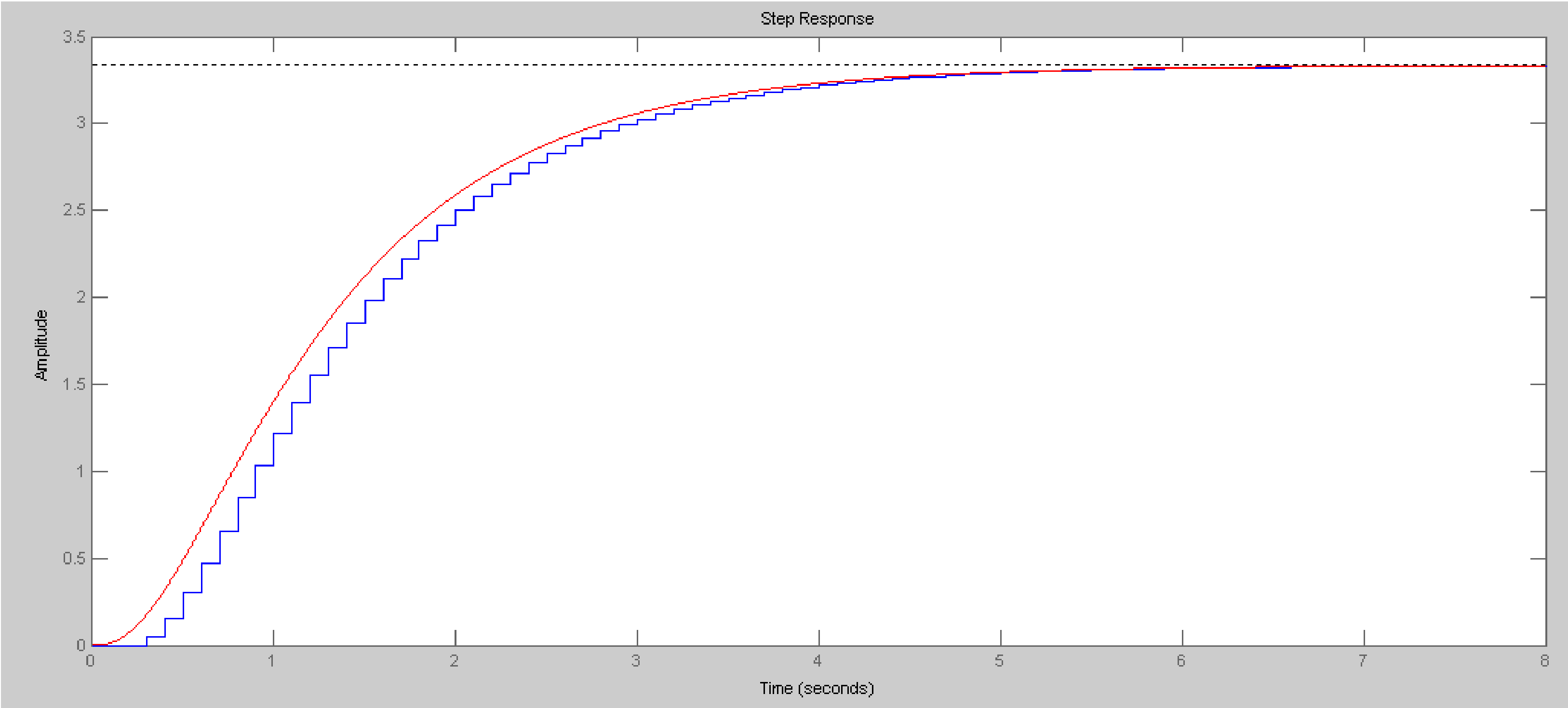
So, G(z) is.

```
Gz = zpk([],[exp(-1*T),exp(-3*T),exp(-10*T)],k,T)

            0.05197
--------------------------------
(z-0.9048) (z-0.7408) (z-0.3679)

Sampling time (seconds): 0.1
```

Checking the answer: Plot the step response of G(z) and G(s)

- Add zeros at z=0 to remove the time delay

```
step(Gz)
hold on
t = [0:0.001:8]';
ys = step(Gs,t);
plot(t,ys,'r');
```



Example 2: Complex Poles

This also works with complex poles and zeros

$$G(s) = \left(\frac{100(s+j5)(s-j5)}{(s+1+j4)(s+1-j4)(s+20)}\right)$$

```
sn = [j*5,-j*5]'
T = 0.1
zn = exp(sns*T)

    0.8775826 - 0.4794255i
    0.8775826 + 0.4794255i

poly(nz)

    1.   - 1.7551651    1.

sd = [-1+j*4,-1-j*4,-20]'

  - 1. - 4.i
  - 1. + 4.i
  - 20.
```

```
zd = exp(sd*T)

    0.8334105 - 0.3523603i
    0.8334105 + 0.3523603i
    0.1353353

poly(zd)

    1.  - 1.8021562    1.0443104  - 0.1108032
```

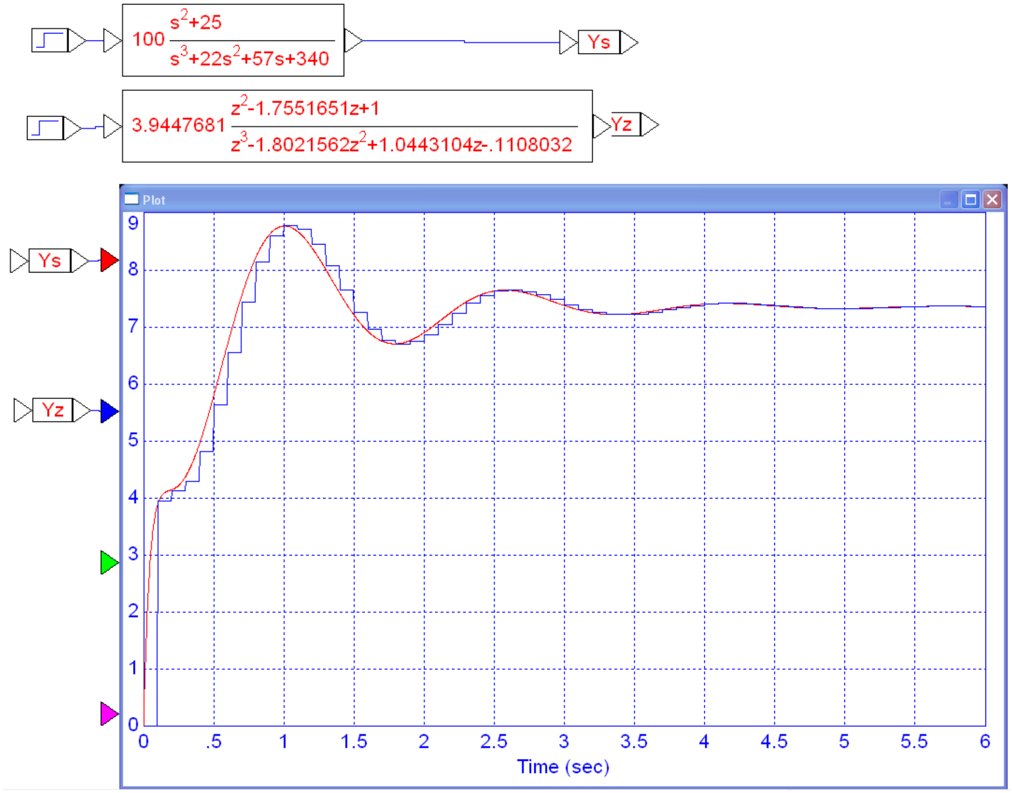
meaning

$$G(z) = k\left(\frac{z^2-1.755z+1.}{z^3-1.802z^2+1.044z-0.110}\right)$$

To find k, match the DC gain:

```
DC = 100*25/340

      7.3529412

k = DC*prod(1-zd)/prod(1-zn)

      3.9447681
```

$$G(z) = 3.944\left(\frac{z^2 - 1.755z + 1.}{z^3 - 1.802z^2 + 1.044z - 0.110}\right)$$



In Matlab: Input the system G(s)

```
z1 = j*5;
z2 = -j*5;
p1 = -1+j*4;
p2 = -1-j*4;
p3 = -2;
Gs = zpk([z1,z2],[p1,p2,p3],100)

   100 (s^2 + 25)
---------------------
(s+2) (s^2 + 2s + 17)
```

Now input G(z). Convert the poles and zeros to the z-plane as e^{sT}.

```
T = 0.1;
Gz = zpk([exp(z1*T),exp(z2*T)],[exp(p1*T),exp(p2*T),exp(p3*T)],1,T)

        (z^2 - 1.755z + 1)
----------------------------------
(z-0.8187) (z^2 - 1.667z + 0.8187)

Sampling time (seconds): 0.1
```

Add a gain, k, to make the DC gains match up:

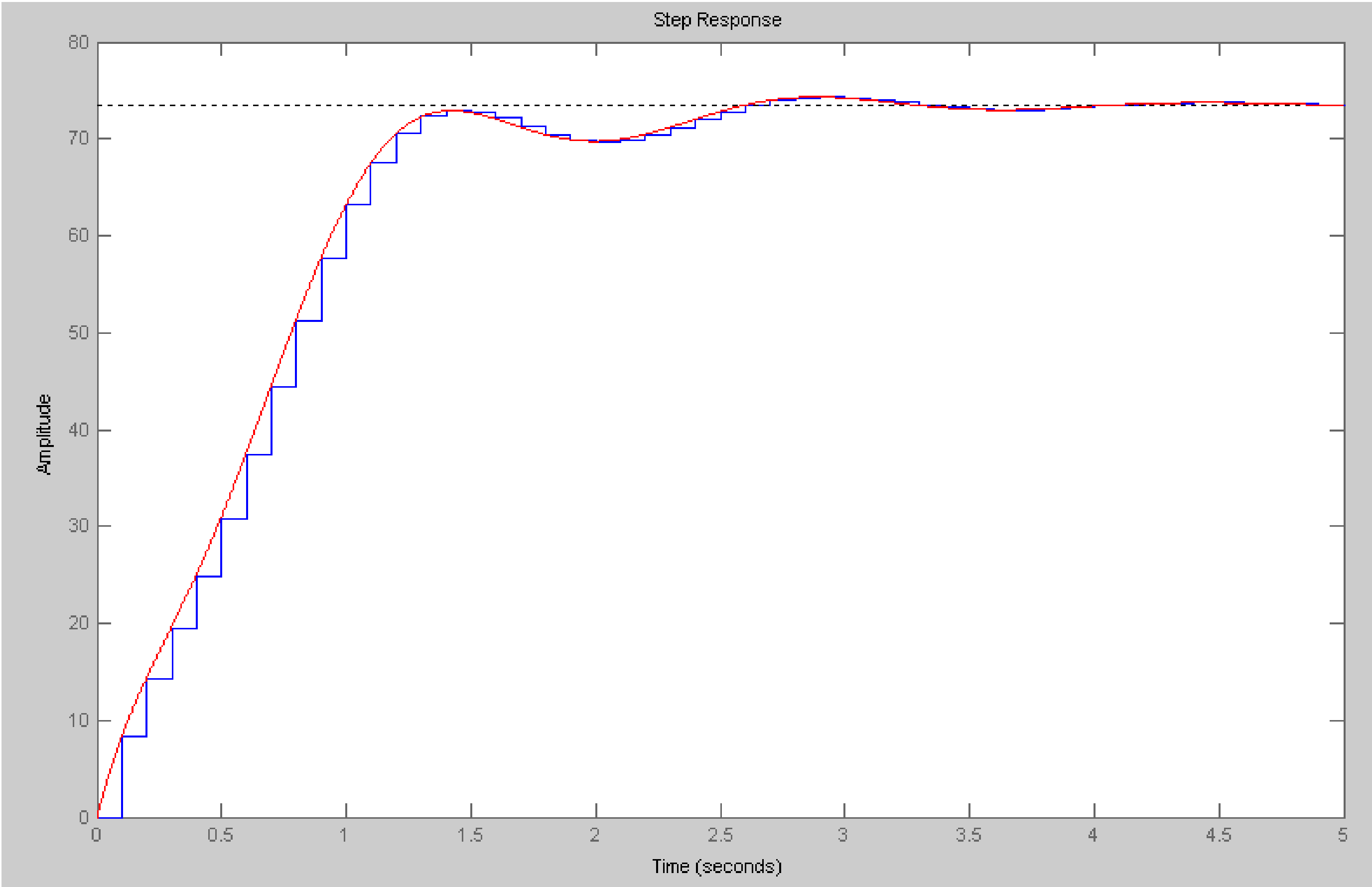
```
DCs = evalfr(Gs,0)

   73.5294

DCz = evalfr(Gz,1)

    8.8913

k = DCs / DCz

    8.2699
```

So, the discrete-time model for G(s) is....

```
Gz = zpk([exp(z1*T),exp(z2*T)],[exp(p1*T),exp(p2*T),exp(p3*T)],k,T)

    8.2699 (z^2 - 1.755z + 1)
----------------------------------
(z-0.8187) (z^2 - 1.667z + 0.8187)

Sampling time (seconds): 0.1
```

Check the result by plotting the step response of G(s) and G(z) on the same graph:

```
step(Gz)
hold on
t = [0:0.001:5]';
ys = step(Gs,t);
plot(t,ys,'r');
```



Handout: Determine the discrete-time equivalent of G(s). Assume T = 0.1 seconds

$$G(s) = \left(\frac{20}{(s+2)(s+5)}\right)$$

Handout: Determine the continuous-time equivalent of G(z). Assume T = 0.1 second

$$G(z) = \left(\frac{0.2z}{(z-0.8)(z-0.6)}\right)$$

Note: Changing the Sampling Rate:

$$z = e^{sT}$$

If you change the sampling rate, G(z) changes

and all of your analysis on G(z) becomes worthless

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+10)}\right)$$

At T = 0.1

$$G(z) \approx \left(\frac{0.051969 \cdot z}{(z-0.9048)(z-0.7408)(z-0.3687)}\right)$$

At T = 0.01

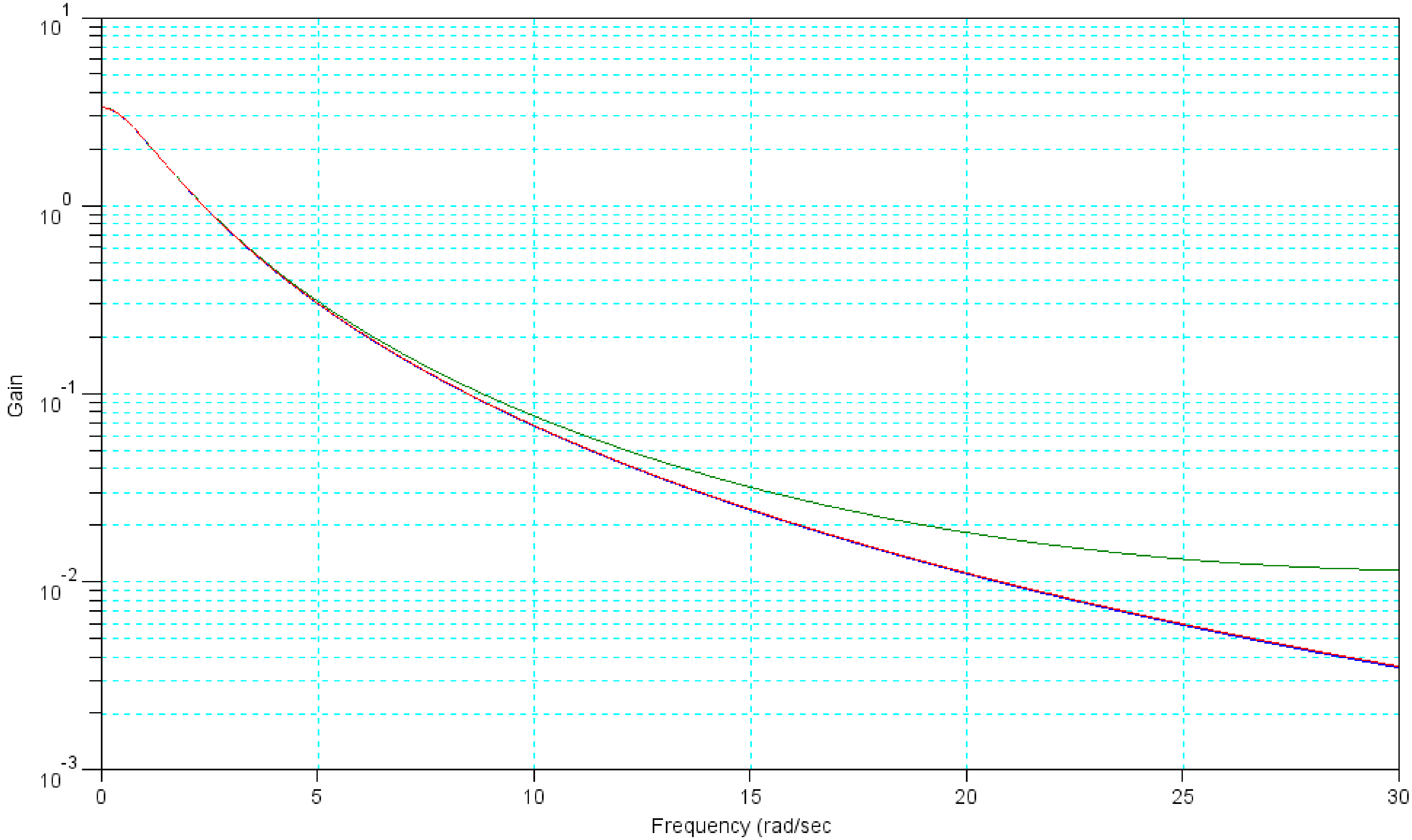
$$G(z) \approx \left(\frac{0.00009393 \cdot z}{(z-0.9900)(z-0.9704)(z-0.9048)}\right)$$

Note 2: Frequency Response of G(z)

If G(s) and G(z) are the same system

- They have the step response
- They have the same frequency response

```
w = [0:0.01:30]';
s = j*w;
Gs = 100 ./ ( (s+1) .* (s+3) .* (s+10) );

T = 0.1;
z = exp(s*T);
Gz = 0.051969*z ./ ( (z-0.9048) .* (z-0.7408) .* (z-0.3687) );

T = 0.01;
z = exp(s*T);
Gz2 = 0.000093938*z ./ ( (z-0.9900).*(z-0.9704).*(z-0.9048) );

plot(w,abs(Gs),w,abs(Gz),w,abs(Gz2));
xlabel('Frequency (rad/sec');
ylabel('Gain');
xgrid(4)
```



Gain of G(s) (blue), G(z) with T = 0.1 (green), and G(z) with T = 0.01 (red)

Summary

There are several ways to convert from G(s) to G(z)

- Substitution:

  Replace '1/s' with a numerical approximation for integration

- Mapping:

  Map the poles and zeros to the z-plane as $z = e^{sT}$
  Then match the DC gain

Either method works. As a check

- Both G(s) and G(z) should have similar step responses
- Both G(s) and G(z) should have similar frequency responses

Time and frequency are related. If one matches, the other will too.