ECE 463/663 - Homework #8

Calculus of Variations. Due Monday, April 1, 2019

Note: If there is no solution to a problem, explain why this is so.

1) Determine the shape of a soap film connecting two rings around the X-axis subject to the constaint

- y(0) = 5
- y(3) = 4

The general solution is of the form

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

Plug in the endpoints to get two equations for two unknowns

$$5 = a\cosh\left(\frac{-b}{a}\right)$$

$$4 = a\cosh\left(\frac{3-b}{a}\right)$$

Solving in Matlab. Create a cost function

```
function [ J ] = cost( z )

   a = z(1);
   b = z(2);

   e1 = a*cosh(b/a) - 5;
   e2 = a*cosh( (3-b)/a ) - 4;

   J = e1.^2 + e2.^2;

   end
```

Solve using fminsearch()

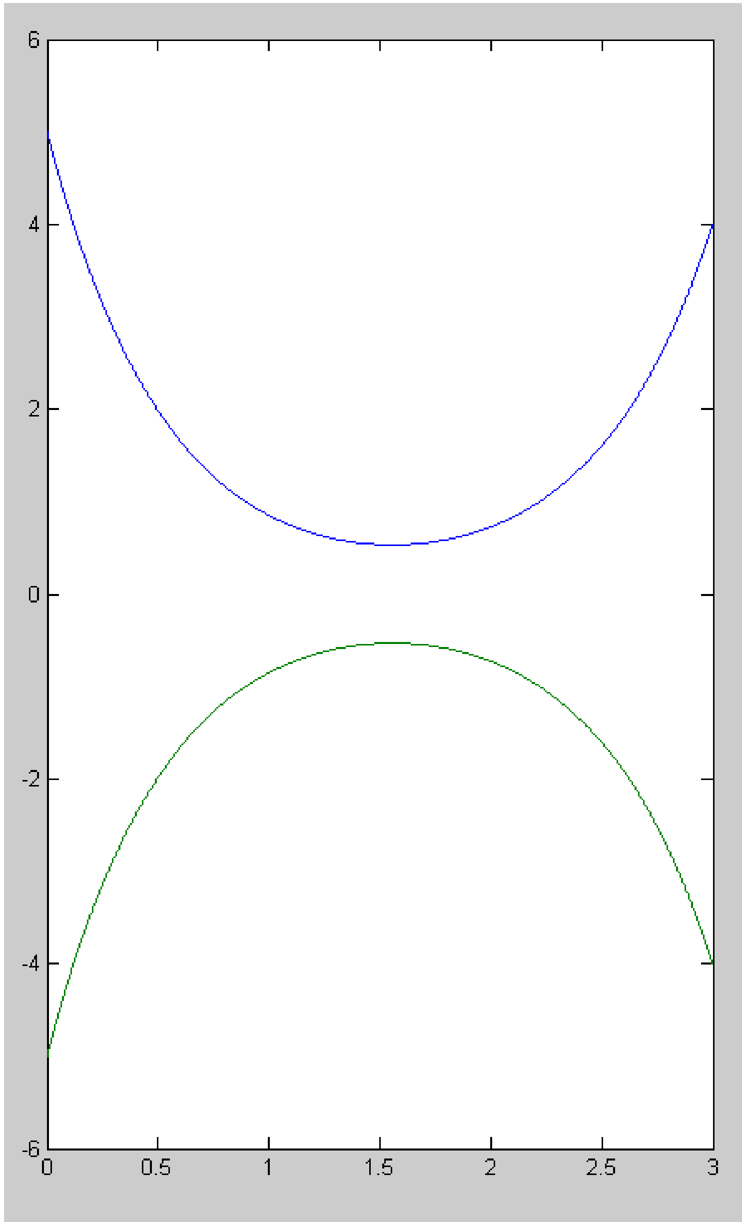
```
>> [z,e] = fminsearch('cost',[1,2])

z =
       a          b
    0.5323    1.5598


e =

  3.8330e-008
```

The sum-squared error (b) is closed to zero, so

ans: $y = 0.5233\cosh\left(\frac{x-1.5598}{0.5323}\right)$

Plotting in Matlab

```
>> x = [0:0.001:3]';
>> y = 0.5323*cosh( (x-1.5598)/0.5323 );
>> plot(x,y,x,-y);
>>
```



Shape of soap film rotated about the X axis

2) Determine the shape of a soap film connecting two rings around the X-axis subject to the constaint

- y(0) = 5
- y(10) = 4

Change the cost function

```
function [ J ] = cost( z )

   a = z(1);
   b = z(2);

   e1 = a*cosh(b/a) - 5;
   e2 = a*cosh( (10-b)/a ) - 4;

   J = e1.^2 + e2.^2;

   end
```

Solve in Matlab

```
>> [z,e] = fminsearch('cost',[1,2])

z =
        a         b
     4.1646    5.2092


e =

    18.7205
```

The sum-squared error is not zero - fmsearch() cannot find a solution.

Turns out there is no solution. The soap film breaks if the two endpoints are too far apart (they were close to breaking in problem #1)

ans: No Solution

3) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 5
- Y(2) = free

The general solution is of the form

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

The left endpoint gives

$$5 = a\cosh\left(\frac{b}{a}\right)$$

The right endpoint gives

$$\dot{y} = \sinh\left(\frac{x-b}{a}\right) = 0$$

$$\sinh\left(\frac{2-b}{a}\right) = 0$$

b = 2

Solving in Matlab

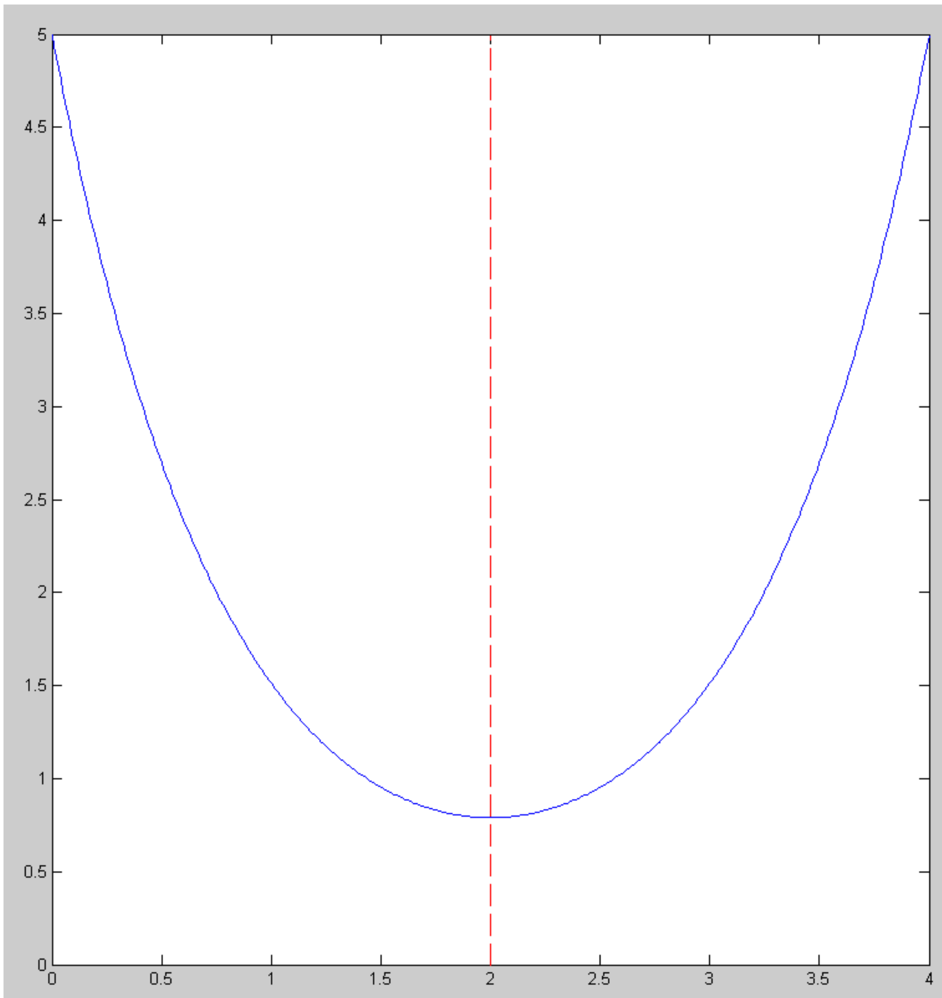
```
function [ J ] = cost4( z )

   a = z(1);
   b = 2;

   e1 = a*cosh( b/a ) - 5;

   J = e1^2;

   end
```

```
[a,e] = fminsearch('cost4',1)

a =  0.7898

e =  9.1019e-008
```

So

$$y = 0.7898\cosh\left(\frac{x-2}{0.7898}\right)$$

```
x = [0:0.01:4]';
a =  z;
b = 2;
y = a*cosh((x-b)/a);
plot(x,y);
plot([2,2],[0,5],'r--')
plot(x,y,'b',[2,2],[0,5],'r--')
```



4) Determine the shape of a hanging chain with gravity in the -y direction. Assume the chain is 10 meters long hanging between two points:

- Left end: x = -3 y = 3
- Right end: x = +3 y = 0

Solution (from lecture notes)

$$y = a\cosh\left(\frac{x-b}{a}\right) - M$$

To solve, you need three equations for three unknowns. Use the two endpoints:

(1) $3 = a\cosh\left(\frac{-3-b}{a}\right) - M$

(2) $0 = a\cosh\left(\frac{3-b}{a}\right) - M$

The third constraint is the length:

$$\left(a\sinh\left(\frac{x-b}{a}\right)\right)_{x_0}^{x_1} = L$$

(3) $a\sinh\left(\frac{3-b}{a}\right) - a\sinh\left(\frac{-3-b}{a}\right) = 10$

Solve in Matlab. Set up a cost function

```
function [ J ] = cost4( z )

   a = z(1);
   b = z(2);
   M = z(3);

   e1 = a*cosh( (-3-b)/a ) - M - 3;
   e2 = a*cosh( (3-b)/a  ) - M;
   e3 = a*sinh( (3-b)/a ) - a*sinh((-3-b)/a) - 10;

   J = e1^2 + e2^2 + e3^2;

   end
```

Solve using fminsearch()

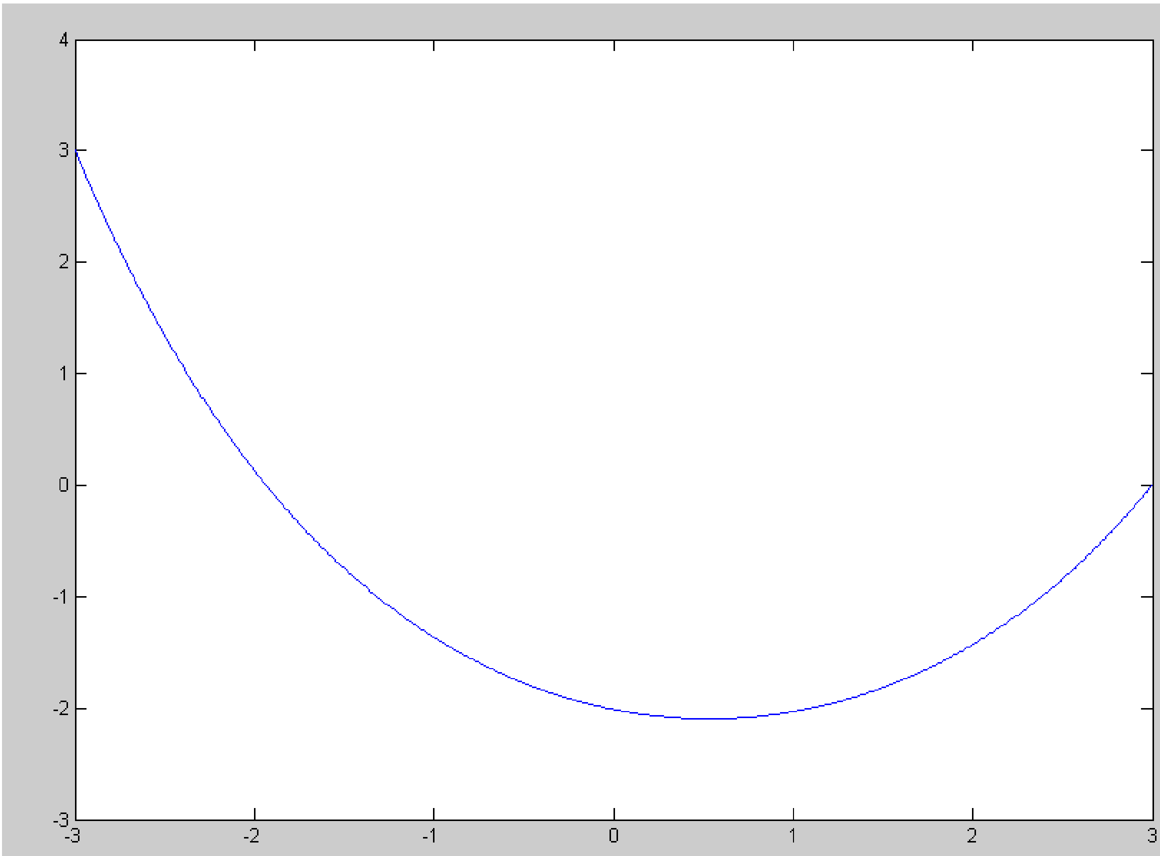
```
>> [z,e] = fminsearch('cost4',10*rand(1,3))

z =    1.7201    0.5324    3.8152

e =  9.6149e-009
```

The sum squared error is almost zero. This is a valid answer

ans: $y = 1.7201\cosh\left(\frac{x-0.5324}{1.7201}\right) - 3.8152$

```
x = [-3:0.01:3]';
a = z(1);
b = z(2);
M = z(3);
y = a*cosh((x-b)/a) - M;
plot(x,y);
```

Plotting in Matlab:



Hanging chain of 10m length connecting points (-3,3) and (+3, 0)

5) Determine x(t) which minimizes the following functional:

$$J = \int_0^2 (x^2 + 5\dot{x}^2)dt$$

subject to the constraints:

$$x(0) = 2$$

$$x(2) = 0$$

Plug into the Euler LaGrange equation

$$F = x^2 + 5\dot{x}^2$$

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \frac{d}{dt}(10\dot{x}) = 0$$

$$2x - 10\ddot{x} = 0$$

Assume all functions are in the form of exp(st)

$$2x - 10s^2 x = 0$$

$$(5s^2 - 1)x = 0$$

Either

- x = 0 (trivial solution), or
- $s = \pm\sqrt{\frac{1}{5}}$

So

$$x(t) = a \cdot e^{0.4472t} + b \cdot e^{-0.4472t}$$

Plug in the two endpoints

$$x(0) = 2 = a + b$$

$$x(2) = 0 = a \cdot e^{0.8944} + b \cdot e^{-0.8944}$$

Solving

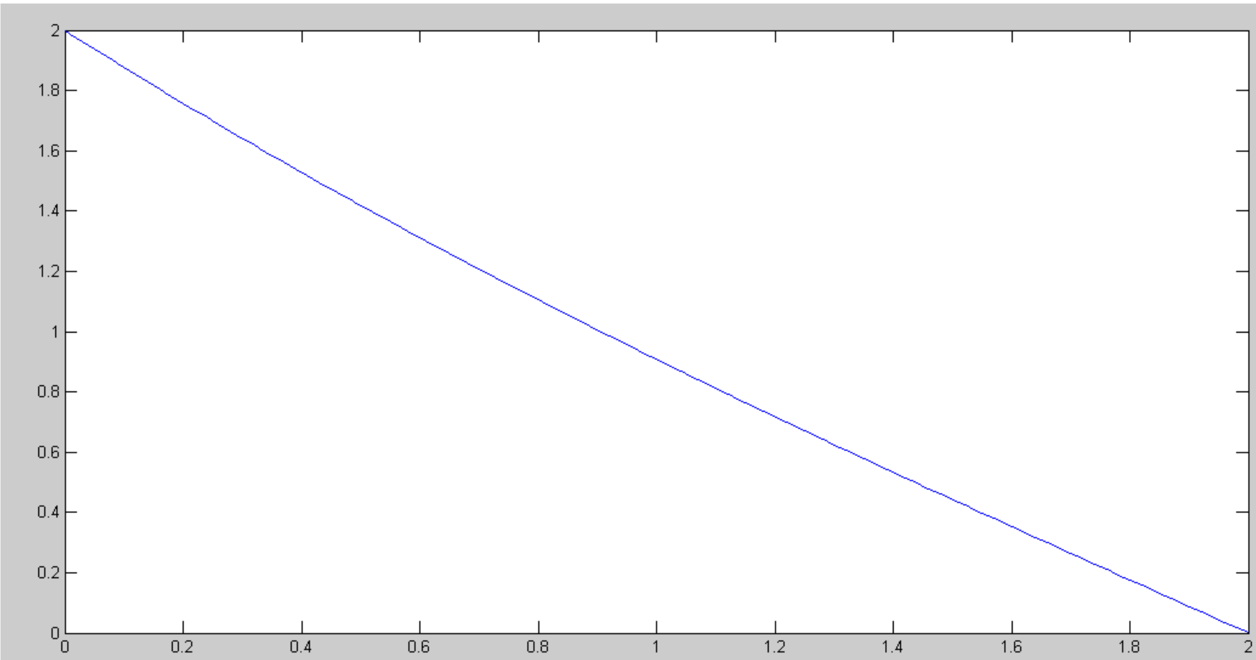
```
X = [1,1; exp(0.8944),exp(-0.8944)]


    1.0000    1.0000
    2.4459    0.4089

inv(X)*[2;0]
```

```
a    -0.4014
b     2.4014
```

$$x(t) = -0.4014 \cdot e^{0.4472t} + 2.4014 \cdot e^{-0.4472t}$$

Plotting in Matlab:

```
t = [0:0.01:2]';
x = -0.4014*exp(0.4472*t) + 2.4014*exp(-0.4472*t);
plot(t,x);
```



6) Determine x(t) which minimizes the following functional:

$$J = \int_0^5 (x^2 + 5u^2)dt$$

subject to the constraints:

$$\dot{x} = -0.1x + u$$

$$x(0) = 2$$

$$x(2) = 0$$

Set up the functional with a LaGrange multiplier:

$$F = x^2 + 5u^2 + M(\dot{x} + 0.1x - u)$$

Solve the three Euler LaGrange equations:

$$F_x - \tfrac{d}{dt}(F_{\dot{x}}) = 0$$

$$F_u - \tfrac{d}{dt}(F_{\dot{u}}) = 0$$

$$F_M - \tfrac{d}{dt}(F_{\dot{M}}) = 0$$

i) $F_x - \tfrac{d}{dt}(F_{\dot{x}}) = 0$

$$(2x + 0.1M) - \tfrac{d}{dt}(M) = 0$$

$$2x + 0.1M - \dot{M} = 0$$

ii) $F_u - \tfrac{d}{dt}(F_{\dot{u}}) = 0$

$$10u - M = 0$$

iii) $F_M - \tfrac{d}{dt}(F_{\dot{M}}) = 0$

$$\dot{x} + 0.1x - u = 0$$

Start with equation ii)

$$M = 10u$$

$$\dot{M} = 10\dot{u}$$

Substitute in to i)

$$2x + u - 10\dot{u} = 0$$

From iii)

$$\dot{x} + 0.1x - u = 0$$

$$u = \dot{x} + 0.1x$$

$$\dot{u} = \ddot{x} + 0.1\dot{x}$$

Substitute

$$2x + u - 10\dot{u} = 0$$

$$2x + (\dot{x} + 0.1x) - 10(\ddot{x} + 0.1\dot{x}) = 0$$

$$-10\ddot{x} + 2.1x = 0$$

$$(s^2 - 0.21)x = 0$$

Either

x = 0 (trivial solution), or

$$s = \pm 0.4583$$

$$x(t) = a \cdot e^{0.4583t} + b \cdot e^{-0.4583t}$$

Plug in the endpoints

$$x(0) = 2 = a + b$$

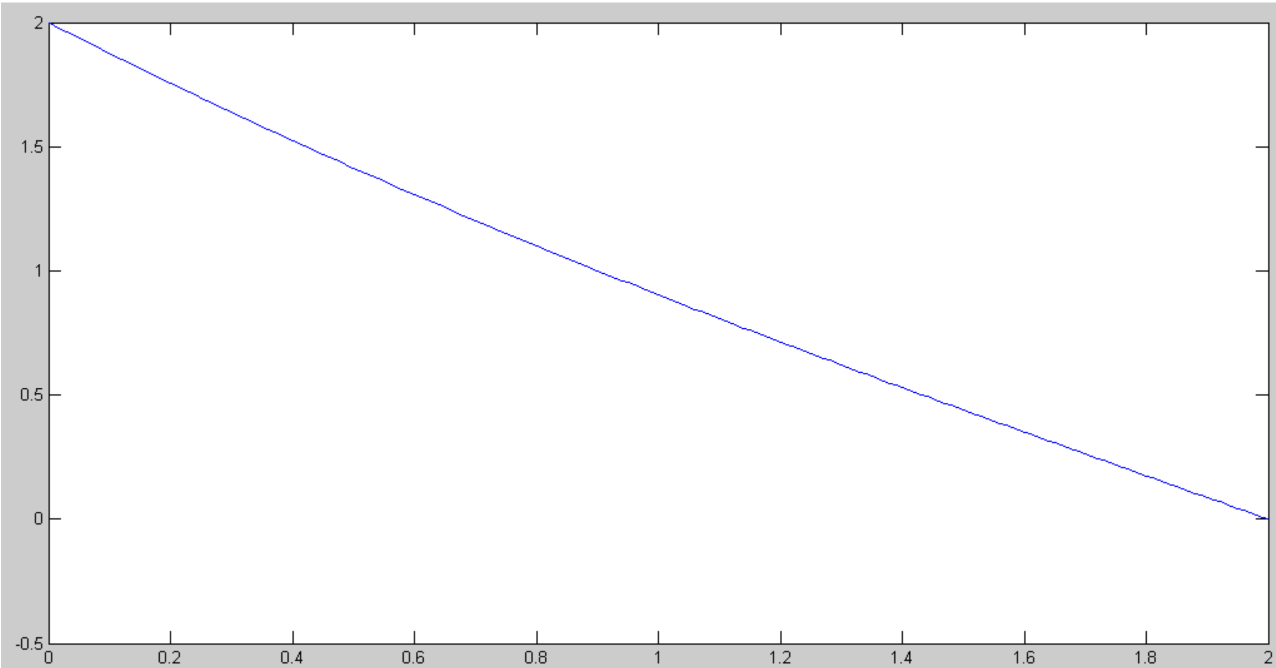
$$x(1) = 0 = a \cdot e^{0.9165} + b \cdot e^{-0.9165}$$

Solving

```
A = [1,1 ; exp(0.9165),exp(-0.9165)]

    1.0000    1.0000
    2.5005    0.3999

ab = inv(A)*[2;0]

a   -0.3808
b    2.3808
```

ans:

$$x(t) = -0.3808 \cdot e^{0.4583t} + 2.3808 \cdot e^{-0.4583t}$$



optimal path for x(t)

LQG Control

7) Cart and Pendulum (HW #5): Design a full-state feedback control law of the form

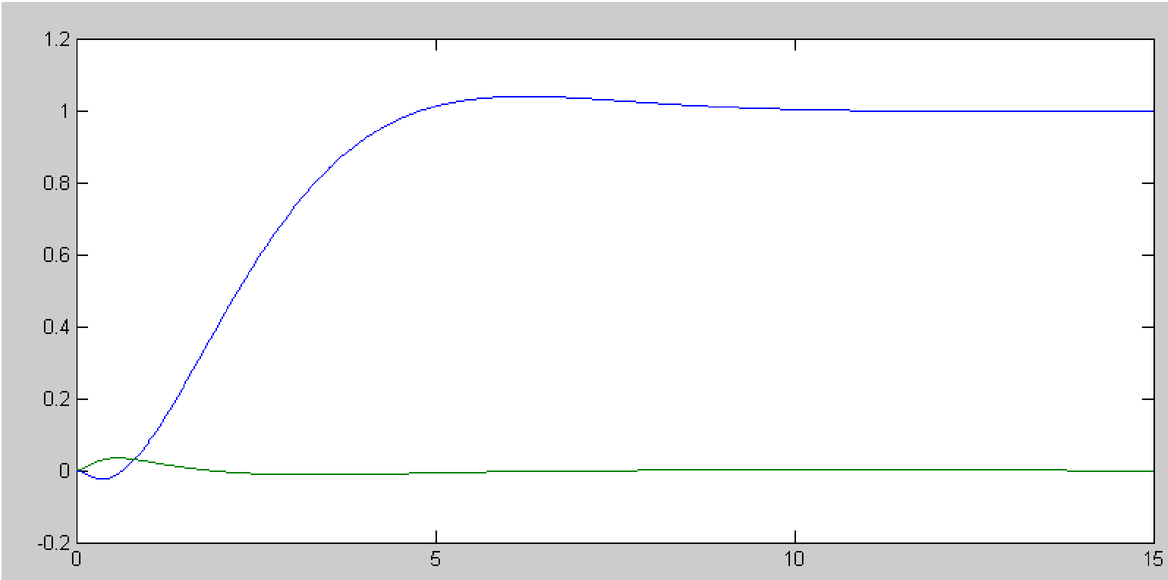
U = KrR - KxX

for the cart and pendulum system from homework #5 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 8 seconds, and
- There is less than 10% overshoot for a step input.

Compare your results with homework #5

```
Kx = lqr(A, B, diag([1,0,0,0]), 1);
eig(A-B*Kx)

  -5.4218 + 0.0615i
  -5.4218 - 0.0615i
  -0.4129 + 0.4036i         too slow
  -0.4129 - 0.4036i


>> Kx = lqr(A, B, diag([3,0,0,0]), 1);
>> eig(A-B*Kx)

  -5.4211 + 0.1065i
  -5.4211 - 0.1065i
  -0.5477 + 0.5266i        fast enough
  -0.5477 - 0.5266i

>> DC = -C*inv(A - B*Kx)*B

DC =   -0.5774

>> Kr = 1/DC

Kr =   -1.7321

>> G = ss(A-B*Kx, B*Kr, Cxq, D);
>> y = step(G, t);
>> plot(t,y);
```



Homework #5: Pole Placement

Pole Location

```
[-0.5 + j*0.682, -0.5-j*0.682, -2, -3]
```

Gains

```
Kx = ppl(A, B, [-0.5 + j*0.682, -0.5-j*0.682, -2, -3])

Kx =   -0.4378  -41.5530   -0.9771   -6.9771
```

LQR

Pole Location

```
eig(A-B*Kx)

  -5.4211 + 0.1065i
  -5.4211 - 0.1065i
  -0.5477 + 0.5266i
  -0.5477 - 0.5266i
```

Gains

```
>> Kx = lqr(A, B, diag([3,0,0,0]), 1)

Kx =   -1.7321  -72.9869   -3.9252  -15.8629
```

8) Ball and Beam (HW #5): Design a full-state feedback control law of the form

U = KrR - KxX

for the ball and beam system from homework #6 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 8 seconds, and
- There is less than 10% overshoot for a step input.

Controller Design:

```
>> A = [0,0,1,0;0,0,0,1;0,-7,0,0;-3.27,0,0,0]
>> B = [0;0;0;0.33]
```

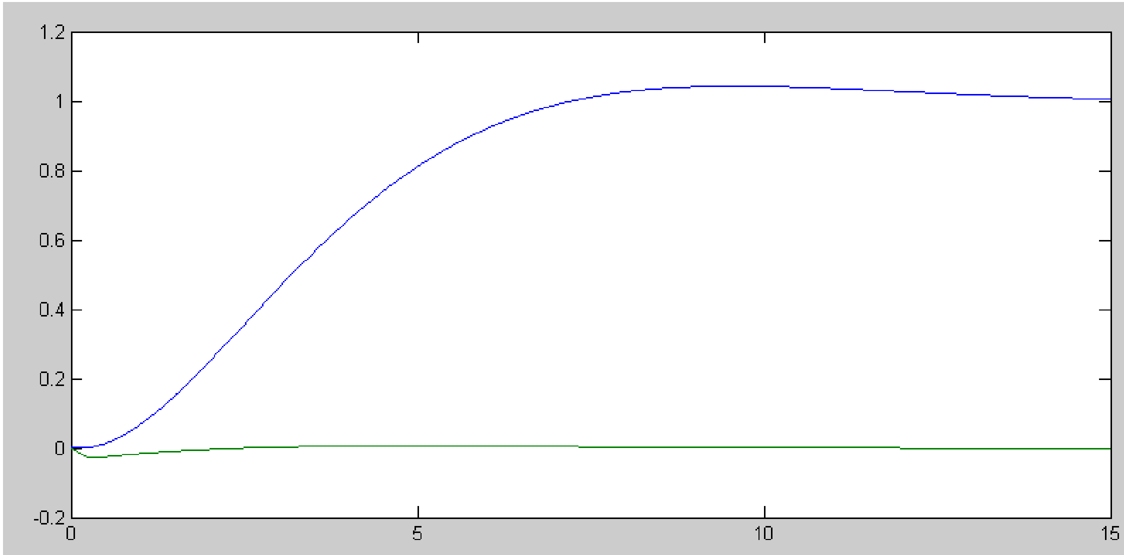
```
Kx = lqr(A, B, diag([1,100000,0,0]), 1);
DC = -C*inv(A - B*Kx)*B;
Kr = 1/DC;
G = ss(A-B*Kx, B*Kr, Cxq, D);
y = step(G, t);
plot(t,y);
```

Adjust Q until the step response looks good

```
Kx =  -19.8685   30.5309   -9.3208   13.6028

Kr =   -9.9594


closed-loop poles

  -7.2158 + 7.2158i
  -7.2158 - 7.2158i
  -0.3324 + 0.3324i
  -0.3324 - 0.3324i
```

Comparing to homework #5

```
Kx = -11.6501 35.1489 -4.1042 18.0018
Kr = -1.8391


closed-loop poles
-0.5 + j*0.682
-0.5-j*0.682
-2
-3
```



Closed-Loop Step Respons of Ball & Beam System