ECE 463/663 - Test #3: Name ______________________

Friday, April 26th Closed Book, Closed Notes. Calculators Permitted

1) Determine the function, y(t), which minimizes the following funcitonal:

$$J = \int_0^{10} (9y^2 + \dot{y}^2) dt$$

subject to the constraint

y(0) = 1

y(10) = 0

$$F_y - \frac{d}{dt}(F_{\dot{y}}) = 0$$

$$(18y) - \frac{d}{dt}(2\dot{y}) = 0$$

$$18y - 2\ddot{y} = 0$$

$$9y - \ddot{y} = 0$$

Assume

$$y = ae^{st}$$

$$9Y - s^2 Y = 0$$

$$(9 - s^2)Y = 0$$

$$s = \{ +3, \ -3 \}$$

meaning

$$y(t) = ae^{3t} + be^{-3t}$$

Plugging in the endpoints

$$y(0) = 1 = a + b$$

$$y(10) = 0 = ae^{30} + be^{-30}$$

Solving

$$a = -0.000000000000000000000000000875 \approx 0$$

$$b = 1.000000000000000000000000000875 \approx 1$$

so

$$\boxed{y(t) \approx 1e^{-3t}}$$

2) Determine the function, y(t), which minimizes the following funcitonal:

$$J = \int_0^2 (9y^2 + u^2) dt$$

subject to the constraints

$y(0) = 1$ $\quad$ $y(10) = 0$ $\quad$ $\dot{y} - 2y = u$

2a) Solve the following Euler LeGrange equation

$$F = (9y^2 + u^2) + m(\dot{y} - 2y - u)$$

$$F_y - \frac{d}{dt}(F_{\dot{y}}) = 0$$

$$(18y - 2m) - \frac{d}{dt}(m) = 0$$

$$\boxed{18y - 2m - \dot{m} = 0}$$

2b) Solve the following Euler LeGrange equation

$$F = (9y^2 + u^2) + m(\dot{y} - 2y - u)$$

$$F_u - \frac{d}{dt}(F_{\dot{u}}) = 0$$

$$2u - m - \frac{d}{dt}(0) = 0$$

$$\boxed{m = 2u}$$

2c) Solve the following Euler LeGrange equation

$$F = (9y^2 + u^2) + m(\dot{y} - 2y - u)$$

$$F_m - \frac{d}{dt}(F_{\dot{m}}) = 0$$

$$\boxed{\dot{y} - 2y - u = 0}$$

2d) Solve for y(t) given the endpoint constraints:

y(0) = 1

y(10) = 0

(1) $$18y - 2m - \dot{m} = 0$$

(2) $$m = 2u$$

(3) $$u = \dot{y} - 2y$$

Substitute (2) in to (1)

(4) $$18y - 4u - 2\dot{u} = 0$$

Substitite (3) into (4)

$$18y - 4(\dot{y} - 2y) - 2(\ddot{y} - 2\dot{y}) = 0$$

$$2\ddot{y} - 26y = 0$$

$$\ddot{y} - 13y = 0$$

$$(s^2 - 13)Y = 0$$

$$s = \pm\sqrt{13}$$

$$y(t) = ae^{\sqrt{13}\,t} + be^{-\sqrt{13}\,t}$$

Plug in the end points

$$y(0) = 1 = a + b$$

$$y(10) = 0 = ae^{36} + be^{-36}$$

Solving

$$a = -0.00000000000000000000000000000000538 \approx 0$$

$$b = 1.00000000000000000000000000000000538 \approx 1$$

or

$$\boxed{y(t) \approx e^{-\sqrt{13}\,t}}$$

3) A road is to be built between two points, A and B. The cost per mile to build the road is proportional to the square of the distance from they axis:

$$J = \int_a^b \left(x^2 \sqrt{1+\dot{y}^2} \right) dx$$

$$F_y - \frac{d}{dx}(F_{\dot{y}}) = 0$$

Find the differential equation which describes the optimal path for the road. (You don't need to solve for y(x), just give the differential equation that y must satisfy.)

$$0 - \frac{d}{dx}\left(\frac{x^2\dot{y}}{\sqrt{1+\dot{y}^2}} \right) = 0$$

(option 1)

$$\frac{x^2\dot{y}}{\sqrt{1+\dot{y}^2}} = c$$ *c is a constant*

$$\boxed{x^2\dot{y} = c\sqrt{1+\dot{y}^2}}$$

(option 2)

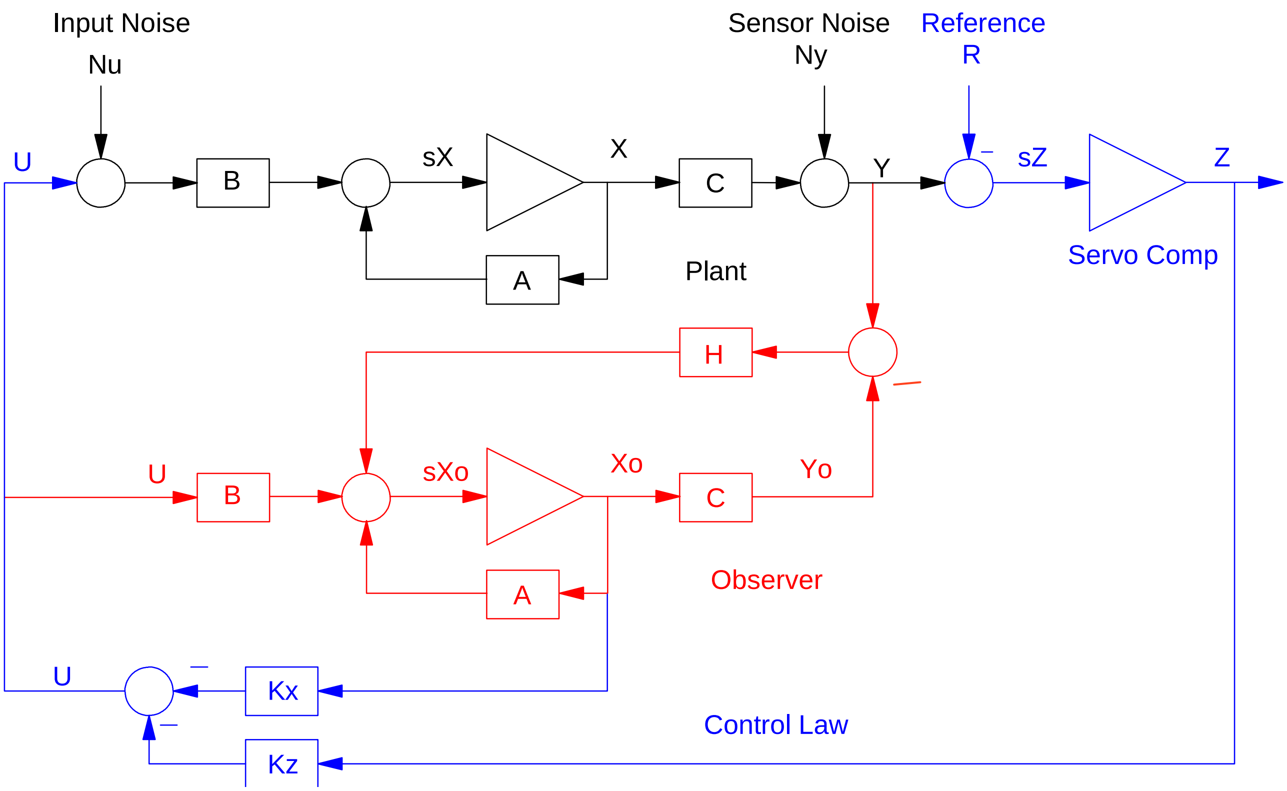
$$\frac{\text{ho de hi - hi de ho}}{\text{ho ho}}$$

$$\frac{\left(\sqrt{1+\dot{y}^2}\right)\left(2x\dot{y}+x^2\ddot{y}\right)-\left(x^2\dot{y}\right)\left(\frac{\dot{y}\ddot{y}}{\sqrt{1+\dot{y}^2}}\right)}{1+\dot{y}^2} = 0$$

$$\boxed{\left(1+\dot{y}^2\right)\left(2x\dot{y}+x^2\ddot{y}\right) - \left(x^2\dot{y}\right)(\dot{y}\ddot{y}) = 0}$$

Whatever the optimal path is, it has to satisfy this differential equation

4) Express the following system in state-space form



$$
\begin{bmatrix} sX \\ sXo \\ sZ \end{bmatrix}
=
\begin{bmatrix} A & -B\,Kx & -B\,Kz \\ H\,C & A - H\,C - B\,Kx & -B\,Kz \\ C & 0 & 0 \end{bmatrix}
\begin{bmatrix} X \\ Xo \\ Z \end{bmatrix}
+
\begin{bmatrix} 0 & B & 0 \\ 0 & 0 & H \\ -1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} R \\ Nu \\ Ny \end{bmatrix}
$$