ECE 463/663 - Homework #3 Solution

Canonical Forms, Similarity Transforms, LaGrangian Dynamics, Controllability. Due Monday, Feb 3rd

Please make the subject "ECE 463 HW#3" if submitting homework electronically to Jacob_Glower@yahoo.com (or on blackboard)

Problem 1-3) For the system

$$Y = \left(\frac{2(s+5)(s+10)}{(s+6)(s+7)(s+8)}\right) U$$

1) Express this system in controller canonical form. (Give the A, B, C, D matrices)

Multiply out

```
>> G = zpk([-5,-10],[-6,-7,-8],2)

Zero/pole/gain:
 2 (s+5) (s+10)
-----------------
(s+6) (s+7) (s+8)

>> tf(G)

Transfer function:
    2 s^2 + 30 s + 100
--------------------------
s^3 + 21 s^2 + 146 s + 336
```

By inspection

$$sX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -336 & -146 & -21 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 100 & 30 & 2 \end{bmatrix} X + [0] U$$

Check

```
A = [0,1,0;0,0,1;-336,-146,-21]

     0     1     0
     0     0     1
  -336  -146   -21

B = [0;0;1];
C = [100,30,2];
D = 0;
G1 = ss(A,B,C,D);
zpk(G1)

 2 (s+10) (s+5)
-----------------
(s+8) (s+7) (s+6)
```

2) Express this system in cascade form

$$Y = \left(\frac{2(s+5)(s+10)}{(s+6)(s+7)(s+8)}\right)U$$

$$Y = \left(\frac{a}{(s+6)(s+7)(s+8)}\right) + \left(\frac{b}{(s+6)(s+7)}\right) + \left(\frac{c}{(s+6)}\right)$$

$$Y = \left(\frac{a}{(s+6)(s+7)(s+8)}\right) + \left(\frac{b(s+8)}{(s+6)(s+7)(s+8)}\right) + \left(\frac{c(s+7)(s+8)}{(s+6)(s+7)(s+8)}\right)$$

Matching terms in the numerator

$$2(s+5)(s+10) = a + b(s+8) + c(s+7)(s+8)$$

$$2s^2 + 30s + 100 = cs^2 + (15c + b)s + (56c + 8b + a)$$

This gives

- c = 2
- b = 0
- a = -12

$$sX = \begin{bmatrix} -6 & 0 & 0 \\ 1 & -7 & 0 \\ 0 & 1 & -8 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 2 & 0 & -12 \end{bmatrix} X + [0]U$$

Check:

```
A = [-6,0,0;1,-7,0;0,1,-8];
B = [1;0;0];
C = [2,0,-12];
D = 0;

G2 = ss(A,B,C,D);
zpk(G2)

 2 (s+5) (s+10)
-----------------
(s+8) (s+7) (s+6)
```

3) Express this system in Jordan (diagonal) form

$$Y = \left(\frac{2(s+5)(s+10)}{(s+6)(s+7)(s+8)} \right) U$$

$$Y = \left(\left(\frac{-4}{s+6} \right) + \left(\frac{12}{s+7} \right) + \left(\frac{-6}{s+8} \right) \right) U$$

Then by inspection

$$sX = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -8 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} -4 & 12 & -6 \end{bmatrix} X + [0] U$$

Checking:

```
A = [-6,0,0 ; 0,-7,0 ; 0,0,-8]

    -6     0     0
     0    -7     0
     0     0    -8

B = [1;1;1];
C = [-4,12,-6];
D = 0;

G3 = ss(A,B,C,D);
zpk(G3)

 2 (s+5) (s+10)
-----------------
(s+8) (s+7) (s+6)
```

4) Assume a system's dynamics are

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -15 & 5 & 0 \\ 5 & -15 & 5 \\ 0 & 5 & -10 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = V_3$$

Express these dynamic with the change in variable

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} V_3 \\ V_2 - 2V_3 \\ V_1 - 5V_2 + 5V_3 \end{bmatrix}$$

In matrix form

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = T^{-1} V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -5 & 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$V = TZ$$

Write the dynamics as

$$sV = AV + BV_0$$

$$Y = CV$$

Sunstituting for V

$$sTZ = ATZ + BV_0$$

$$Y = CTZ$$

or

$$sZ = (T^{-1}AT)Z + (T^{-1}B)V_0 = A_z Z + B_z V_0$$

$$Y = CTZ = C_z Z$$

In Matlab

```
A = [-15,5,0 ; 5,-15,5 ; 0,5,-10]
B = [5;0;0];
C = [0,0,1];
D = 0;

Gx = ss(A,B,C,D);

            125
----------------------------
(s+21.23) (s+12.77) (s+5.99)

Ti = [0,0,1 ; 0,1,-2 ; 1,-5,5]

     0     0     1
     0     1    -2
     1    -5     5

T = inv(Ti)

     5     5     1
     2     1     0
     1     0     0

Az = inv(T)*A*T

     0     5     0
     0     0     5
   -65   -95   -40

Bz = inv(T)*B

     0
     0
     5

Cz = C*T

     1     0     0

Dz = D

     0


Gz = ss(Az, Bz, Cz, Dz);

            125
----------------------------
(s+21.23) (s+12.77) (s+5.99)
```

$$sZ = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ -65 & -95 & -40 \end{bmatrix} Z + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} Z + [0] U$$

Note that this is almost controller canonical form

LaGrangian Dynamics

A 1kg ball is rolling in a bowl with the shape

$$y = 2\cosh(x) - 2 = e^{x} + e^{-x} - 2$$

note:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

6) Determine the kinetic and potential energy of this ball as a function of x: Gravity is in the -y direction.

$$x = x \qquad \dot{x} = \dot{x}$$

$$y = 2\cosh(x) \qquad \dot{y} = 2\sinh(x)\dot{x}$$

Potential Energy

$$PE = mgy = 2g\cosh(x)$$

Kinetic Energy (asuming a solid sphere)

$$KE = 0.7m(\dot{x}^2 + \dot{y}^2)$$

$$KE = 0.7\left(\dot{x}^2 + (2\sinh(x)\dot{x})^2\right)$$

$$KE = 0.7\left(1 + 4\sinh^2(x)\right)\dot{x}^2$$

7) Determine the dynamics for this ball as it rolls in the bowl

The LaGrangian is then

$$L = KE - PE$$

$$L = 0.7\left(1 + 4\sinh^2(x)\right)\dot{x}^2 - 2g\cosh(x)$$

The force on the ball is then

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(1.4\left(1 + 4\sinh^2(x)\right)\dot{x}\right) - (5.6\sinh(x)\cosh(x)\dot{x}^2 - 2g\sinh(x))$$

$$F = 1.4\left(1 + 4\sinh^2(x)\right)\ddot{x} + 11.2\sinh(x)\cosh(x)\dot{x}^2$$

$$-(5.6\sinh(x)\cosh(x)\dot{x}^2 - 2g\sinh(x))$$

Simplifying

$$F = 1.4\left(1 + 4\sinh^2(x)\right)\ddot{x} + 5.6\sinh(x)\cosh(x)\dot{x}^2 + 2g\sinh(x)$$

Assuming F = 0

$$\ddot{x} = -\frac{5.6\sinh(x)\cosh(x)\dot{x}^2 + 2g\sinh(x)}{1.4\left(1 + 4\sinh^2(x)\right)}$$

To simulate, the distance the ball has rolled is

$$d = \int \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\dot{d} = \sqrt{\dot{x}^2 + (2\sinh(x)\dot{x})^2}$$

$$\dot{d} = \dot{x}\sqrt{1 + 4\sinh^2(x)}$$

The angle of the ball (for simulation) is

$$d = r\theta$$