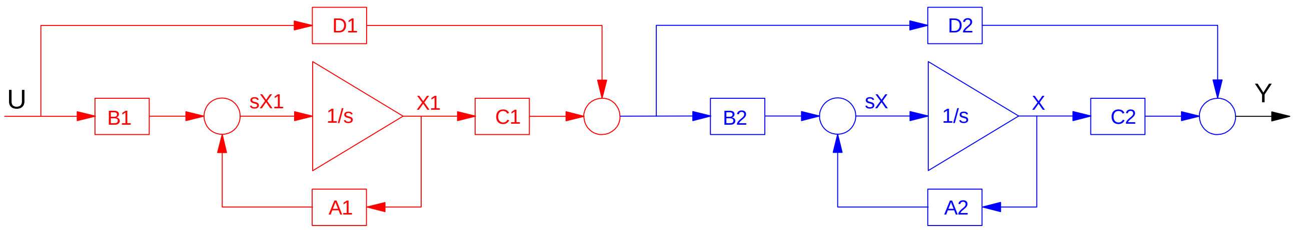
ECE 463/663 - Homework #4

Block Diagrams and LaGrangian Dynamics. Due Monday, February 10th

1) Determine the state-space model for two systems cascaded together:



$$sX_1 = A_1X_1 + B_1U$$

$$sX_2 = A_2X_2 + B_2C_1X_1 + B_2D_1U$$

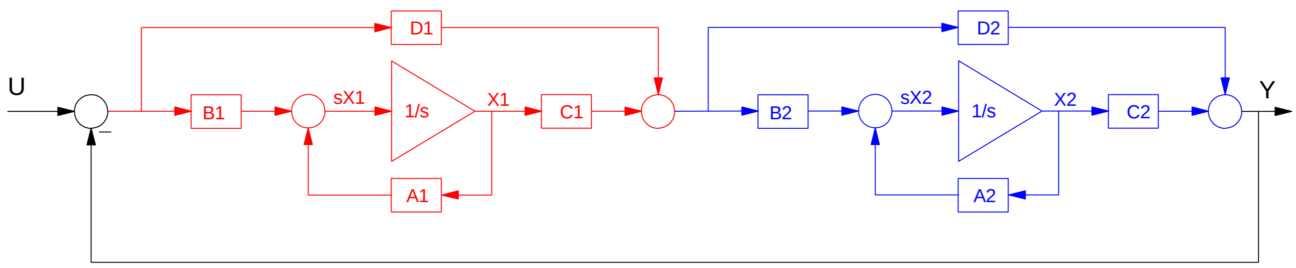
$$Y = C_2X_2 + D_2C_1X_1 + D_2D_1$$

In state-space form

$$\begin{bmatrix} sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}U$$

$$Y = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [D_2D_1]U$$

2) Determine the state-space model for two systems in a feedback configuration:



$$sX_1 = A_1X_1 + B_1E$$

$$sX_2 = A_2X_2 + B_2C_1X_1 + B_2D_1E$$

$$E = U - (C_2X_2 + D_2C_1X_1 + D_2D_1E)$$

$$Y = C_2X_2 + D_2C_1X_1 + D_2D_1E$$

Solving for E

$$(1 + D_2D_1)E = U - C_2X_2 - D_2C_1X_1$$

$$E = (1 + D_2D_1)^{-1}(U - C_2X_2 - D_2C_1X_1)$$

Plugging into the sX1 and sX2 equations:

$$sX_1 = A_1X_1 + B_1(1 + D_2D_1)^{-1}(U - C_2X_2 - D_2C_1X_1)$$

$$sX_2 = A_2X_2 + B_2C_1X_1 + B_2D_1(1 + D_2D_1)^{-1}(U - C_2X_2 - D_2C_1X_1)$$

$$Y = C_2\left(1 - \frac{D_2D_1}{1+D_2D_1}\right)X_2 + D_2C_{12}\left(1 - \frac{D_2D_1}{1+D_2D_1}\right)X_1 + \left(\frac{D_2D_1}{1+D_2D_1}\right)U$$

Grouping terms

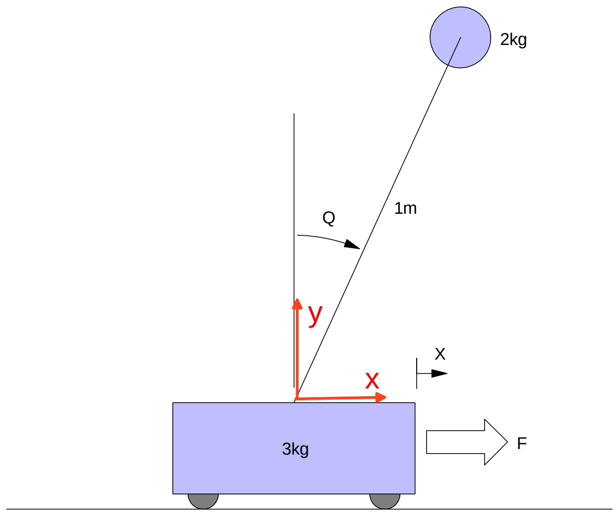
$$\begin{bmatrix} sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} \left(A_1 - \frac{B_1D_2C_1}{1+D_2D_1}\right) & -\left(\frac{B_1C_2}{1+D_2D_1}\right) \\ \left(B_2C_1 - \frac{B_2D_1D_2C_1}{1+D_2D_1}\right) & \left(A_2 - \frac{B_2D_1C_2}{1+D_2D_1}\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \left(\frac{B_1}{1+D_2D_1}\right) \\ \left(\frac{B_2D_1}{1+D_2D_1}\right) \end{bmatrix} U$$

$$Y = \begin{bmatrix} D_2C_{12}\left(1 - \frac{D_2D_1}{1+D_2D_1}\right) & C_2\left(1 - \frac{D_2D_1}{1+D_2D_1}\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \left[\left(\frac{D_2D_1}{1+D_2D_1}\right)\right] U$$

(over)

3) (30 pt) Derive the dynamics for an inverted pendulum where

- m1 = 2kg (mass of ball)
- m2 = 3kg (mass of cart)
- L = 1.0m (length of arm)



First, determine the potential energy (PE) and kinetic energy (KE) for this system

Cart (mass 1)

$$PE_1 = 0$$

$$KE_1 = \tfrac{1}{2} \cdot 3kg \cdot \dot{x}^2$$

Ball (mass 2)

$$x_2 = x + \sin\theta \qquad y_2 = \cos\theta$$

$$\dot{x}_2 = \dot{x} + \cos(\theta)\dot{\theta} \qquad \dot{y}_2 = -\sin(\theta)\dot{\theta}$$

$$PE_2 = m_2 g y_2$$

$$PE_2 = 2 \cdot g \cdot \cos\theta$$

$$KE_2 = \tfrac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$KE_2 = \tfrac{1}{2} \cdot 2 \cdot \left(\left(\dot{x} + \cos(\theta)\dot{\theta}\right)^2 + \left(-\sin(\theta)\dot{\theta}\right)^2\right)$$

$$KE_2 = \left(\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \cos^2(\theta)\dot{\theta}^2 + \sin^2(\theta)\dot{\theta}^2\right)$$

$$KE_2 = \left(\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \dot{\theta}^2\right)$$

The LaGrangian is

$$L = KE - PE$$

$$L = (KE_1 + KE_2) - (PE_1 + PE_2)$$

$$L = (1.5\dot{x}^2) + \left(\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \dot{\theta}^2\right) - (2 \cdot g \cdot \cos\theta)$$

$$\boxed{L = 2.5\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \dot{\theta}^2 - 2g\cos\theta}$$

Take the partials:

$$L = 2.5\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \dot{\theta}^2 - 2g\cos\theta$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}\left(5\dot{x} + 2\cos(\theta)\dot{\theta}\right) - (0)$$

$$F = 5\ddot{x} - 2\sin(\theta)\dot{\theta}^2 + 2\cos(\theta)\ddot{\theta}$$

$$L = 2.5\dot{x}^2 + 2\dot{x}\cos(\theta)\dot{\theta} + \dot{\theta}^2 - 2g\cos\theta$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \tfrac{d}{dt}\left(2\dot{x}\cos(\theta) + 2\dot{\theta}\right) - \left(-2\dot{x}\sin(\theta)\dot{\theta} + 2g\sin\theta\right)$$

$$T = 2\ddot{x}\cos(\theta) - 2\dot{x}\sin(\theta)\dot{\theta} + 2\ddot{\theta} - \left(-2\dot{x}\sin(\theta)\dot{\theta} + 2g\sin\theta\right)$$

$$T = 0 = 2\ddot{x}\cos(\theta) + 2\ddot{\theta} - 2g\sin\theta$$

Grouping terms

$$5\ddot{x} + 2\cos(\theta)\ddot{\theta} = F + 2\sin(\theta)\dot{\theta}^2$$

$$2\ddot{x}\cos(\theta) + 2\ddot{\theta} = 2g\sin\theta$$

Putting these in matrix form

$$\begin{bmatrix} 5 & 2\cos\theta \\ 2\cos\theta & 2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2\sin(\theta)\dot{\theta}^2 \\ 2g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

The linearized dynamics at the point

$$\begin{bmatrix} 5 & 2\cos\theta \\ 2\cos\theta & 2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2\sin(\theta)\dot{\theta}^2 \\ 2g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 2g\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 2g\theta \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -6.533\theta \\ 16.333\theta \end{bmatrix} + \begin{bmatrix} 0.333 \\ -0.333 \end{bmatrix} F$$

In state-space form

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -6.533 & 0 & 0 \\ 0 & 16.333 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.333 \\ -0.333 \end{bmatrix} F$$

The eigenvalues are { 0, 0, +4.04, -4.04 }

If you reverse gravity, you get a gantry system

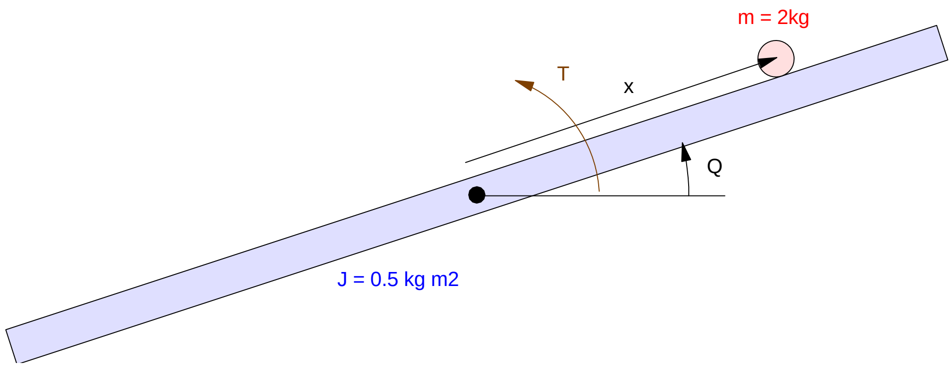
$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6.533 & 0 & 0 \\ 0 & -16.333 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.333 \\ -0.333 \end{bmatrix} F$$

The eigenvalues are { 0, 0, j4.04, -j4.04}

4) (30pt) Derive the dynamics for a ball and beam system where

- J = 0.5 kg m2 (the inertia of the beam)
- m = 2kg (the mass of the ball)

Find the linearized dynamics at $(\theta = 0^0, x = 1.0m)$



First, determine the kinetic and potential energy in the system

Beam:

$$PE_1 = 0$$

$$KE_1 = \tfrac{1}{2}J\dot{\theta}^2 = 0.25\dot{\theta}^2$$

Ball:

$$x_2 = r\cos\theta \qquad \dot{x}_2 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$y_2 = r\sin\theta \qquad \dot{y}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$PE_2 = m_2 g y_2$$

$$PE_2 = 2gr\sin\theta$$

KE (assume a solid sphere)

$$KE_2 = \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \tfrac{1}{5}m_2\dot{r}^2$$

$$KE_2 = \left(\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2\right) + \tfrac{2}{5}\dot{r}^2$$

$$KE_2 = \left(\dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2\dot{r}r\dot{\theta}\sin\theta\cos\theta\right) +$$

$$+\left(\dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2\dot{r}r\dot{\theta}\sin\theta\cos\theta\right) + \tfrac{2}{5}\dot{r}^2$$

$$KE_2 = \dot{r}^2 + r^2\dot{\theta}^2 + \tfrac{2}{5}\dot{r}^2$$

$$KE_2 = \tfrac{7}{5}\dot{r}^2 + r^2\dot{\theta}^2$$

The LaGrangian is then

$$L = (KE_1 + KE_2) - (PE_1 + PE_2)$$

$$L = \left(0.25\dot{\theta}^2 + \tfrac{7}{5}\dot{r}^2 + r^2\dot{\theta}^2\right) - (0 + 2gr\sin\theta)$$

$$L = 0.25\dot{\theta}^2 + 1.4\dot{r}^2 + r^2\dot{\theta}^2 - 2gr\sin\theta$$

Taking partials

$$L = 0.25\dot{\theta}^2 + 1.4\dot{r}^2 + r^2\dot{\theta}^2 - 2gr\sin\theta$$

$$F = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$0 = \tfrac{d}{dt}(2.8\dot{r}) - \left(2r\dot{\theta}^2 - 2g\sin\theta\right)$$

$$0 = 2.8\ddot{r} - 2r\dot{\theta}^2 + 2g\sin\theta$$

$$L = 0.25\dot{\theta}^2 + 1.4\dot{r}^2 + r^2\dot{\theta}^2 - 2gr\sin\theta$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \tfrac{d}{dt}\left(0.5\dot{\theta} + 2r^2\dot{\theta}\right) - (-2gr\cos\theta)$$

$$T = 0.5\ddot{\theta} + 4r\dot{r}\dot{\theta} + 2r^2\ddot{\theta} + 2gr\cos\theta$$

grouping terms

$$2.8\ddot{r} = 2r\dot{\theta}^2 - 2g\sin\theta$$

$$(0.5 + 2r^2)\ddot{\theta} = T - 4r\dot{r}\dot{\theta} - 2gr\cos\theta$$

In matrix form

$$\begin{bmatrix} 2.8 & 0 \\ 0 & (0.5 + 2r^2) \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2r\dot{\theta}^2 - 2g\sin\theta \\ -4r\dot{r}\dot{\theta} - 2gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearizing at r = 1.0, $\theta = 0$

$$\begin{bmatrix} 2.8 & 0 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -2g\theta \\ -2gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

$$\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -7\theta \\ -7.84r \end{bmatrix} + \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} T$$

In state-space form

$$s \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -7.84 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix} T$$

The eigenvalues are { - 2.722, +2.722, +j2.722, -j2.722 }