ECE 463/663 - Homework #8

Calculus of Variations. LQG Control. Due Monday, April 6th

Soap Film

1) Calculate the shape of a soap film connecting two rings around the X axis:
   - Y(0) = 10
   - Y(5) = 10

From the lecture notes, the surface area of a soap film is

$$J = \int y\sqrt{1+\dot{y}^2}\,dx$$

The minimum comes from the Euler LeGrange equation, resulting in

$$y = a\ \cosh\left(\frac{x-b}{a}\right)$$

Plugging in the two end points

$$y(0) = 10 = a\cosh\left(\frac{b}{a}\right)$$

$$y(5) = 10 = a\cosh\left(\frac{5-b}{a}\right)$$

Solve two equations for two unknowns. Create a function in Matlab

```
function [ J ] = cost_soap( z )

   a = z(1);
   b = z(2);

   e1 = a * cosh(b/a) - 10;
   e2 = a * cosh((5-b)/a) - 10;

   J = e1^2 + e2^2;

   end
```

Call it using *fminsearch*

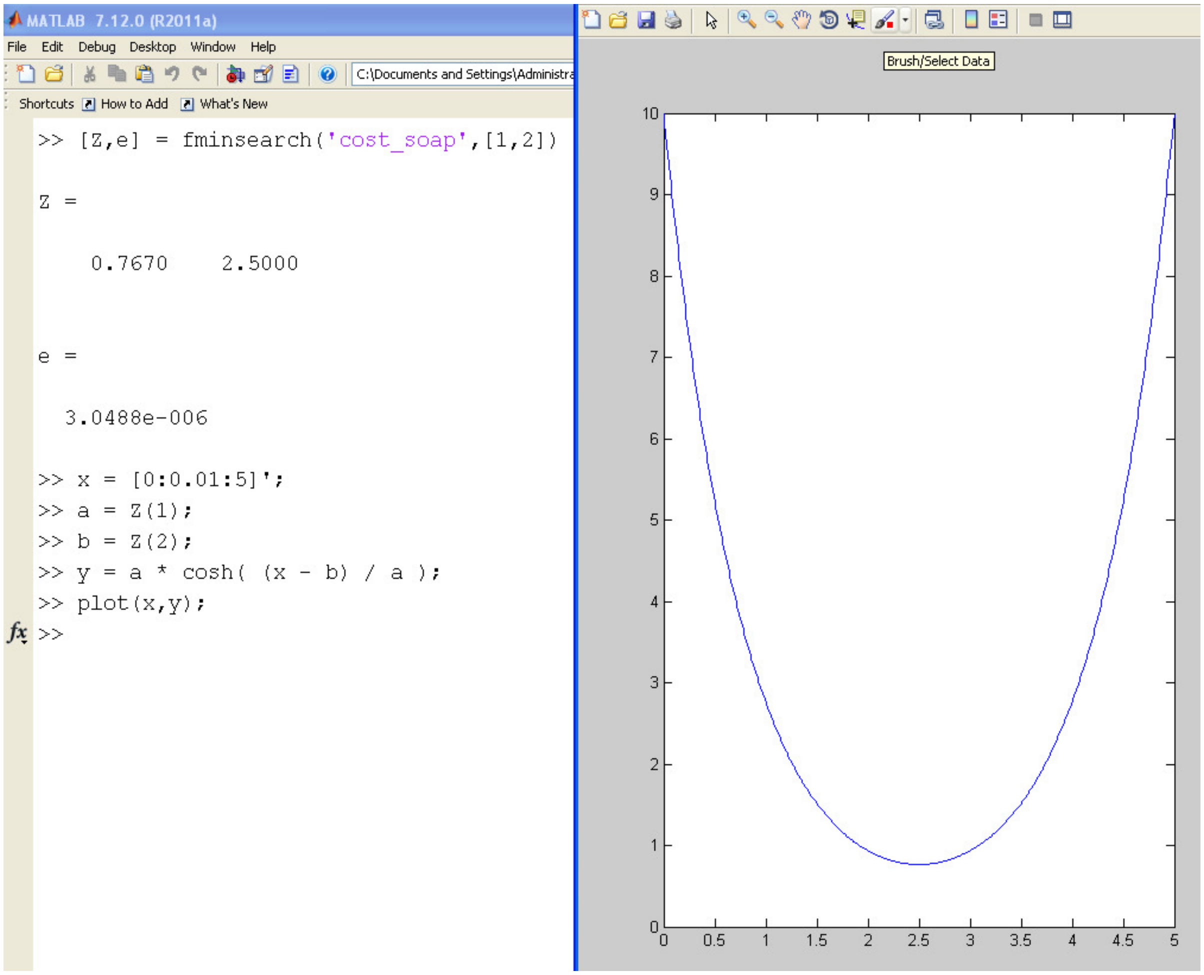
```
>> [Z,e] = fminsearch('cost_soap',[1,2])

              a           b
Z =      0.7670      2.5000

e =      3.0488e-006
```

$$y = 0.7670\ \cosh\left(\frac{x-2.5}{0.7670}\right)$$

```
>>
```


MATLAB 7.12.0 (R2011a)
File Edit Debug Desktop Window Help
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Shortcuts How to Add What's New
>> [Z,e] = fminsearch('cost_soap',[1,2])
Z =
0.7670 2.5000
e =
3.0488e-006
>> x = [0:0.01:5]';
>> a = Z(1);
>> b = Z(2);
>> y = a * cosh((x - b) / a);
>> plot(x,y);
>>
Brush/Select Data
0
1
2
3
4
5
6
7
8
9
10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

2) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 10
- Y(2) = free

This results in

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

$$y(0) = 10 = a\cosh\left(\frac{b}{a}\right)$$

The constraint at the right endpoint is

$$y' = 0 = \sinh\left(\frac{x-b}{a}\right)$$

$$y'(2) = 0 = \sinh\left(\frac{2-b}{a}\right)$$

Setting up a cost function in matlab

```
function [ J ] = cost_soap( z )

  a = z(1);
  b = z(2);

  e1 = a * cosh(b/a) - 10;
  e2 = sinh((2-b)/a);

  J = e1^2 + e2^2;

  end
```

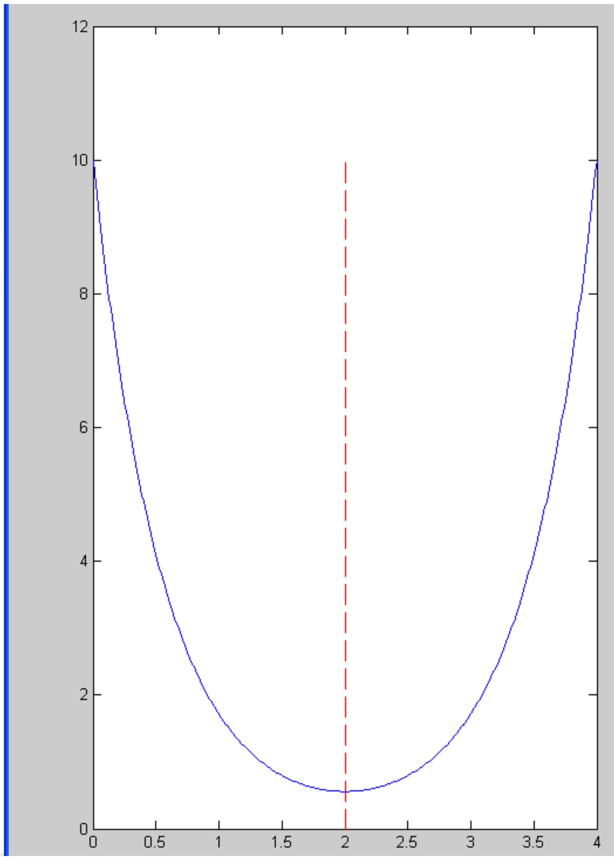
Solving

```
>> [Z,e] = fminsearch('cost_soap',[1,2])

Z =     0.5593     2.0000

e =   5.6278e-009
```

resulting in

$$y = 0.5593\cosh\left(\frac{x-2}{0.5593}\right)$$

```
e =

  3.0488e-006

>> x = [0:0.01:5]';
>> a = Z(1);
>> b = Z(2);
>> y = a * cosh( (x - b) / a );
>> plot(x,y);
>> [Z,e] = fminsearch('cost_soap',[1,2])

Z =

    0.5593    2.0000


e =

  5.6278e-009

>> x = [0:0.01:4]';
>> a = Z(1);
>> b = Z(2);
>> y = a * cosh( (x - b) / a );
>> plot(x,y);
>> plot([2,2],[0,10],'r--')
>> plot(x,y,'b',[2,2],[0,10],'r--')
>>
```



Hanging Chain

3) Calculate the shape of a hanging chain subject to the following constraints

- Length of chain = 11 meters
- Left Endpoint: (0,5)
- Right Endpoint: (10,5)

The functional is

$$F = x\sqrt{1+\dot{y}^2}\,dx + M\sqrt{1+\dot{y}^2}$$

which has the solution of

$$y = a\cosh\left(\tfrac{x-b}{a}\right) - M$$

along with the constraint

$$\left(a\sinh\left(\tfrac{x-b}{a}\right)\right)_0^{10} = L$$

This gives three equations and three unknowns:

$$y(0) = 5 = a\cosh\left(\tfrac{0-b}{a}\right) - M$$

$$y(10) = 5 = a\cosh\left(\tfrac{10-b}{a}\right) - M$$

$$a\sinh\left(\tfrac{10-b}{a}\right) - a\sinh\left(\tfrac{0-b}{a}\right) = 11$$

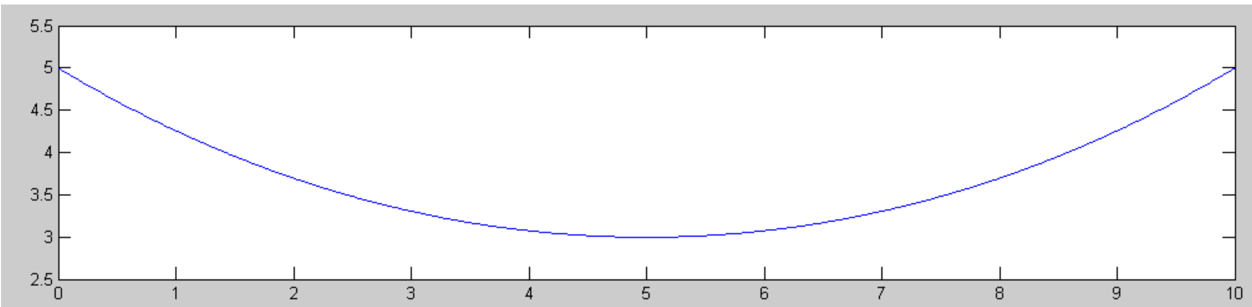
Solving in Matlab

```
function [ J ] = cost_chain( z )

   a = z(1);
   b = z(2);
   M = z(3);

   e1 = a * cosh(b/a) - M - 5;
   e2 = a * cosh((10-b)/a) - M - 5;
   e3 = a*sinh((10-b)/a) - a*sinh((0-b)/a) - 11;

   J = e1^2 + e2^2 + e3^2;

   end
```

```
>> [Z,e] = fminsearch('cost_chain',[1,2,3])

          a           b          M
Z =    6.5497     5.0000     3.5527

e =   2.6083e-010
```

meaning

$$y = 6.5497\cosh\left(\frac{x-5}{6.5497}\right) - 3.5527$$



Ricatti Equation

4) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^8 (x^2 + 100\dot{x}^2)dt$$

x(0) = 5

x(8) = 4

$$F = x^2 + 100\dot{x}^2$$

The Euler LeGrange equation is

$$\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\frac{d}{dt}(200\dot{x}) - (2x) = 0$$

$$100\ddot{x} - x = 0$$

$$(100s^2 - 1)x = 0$$

Either

- x = 0 *trivial solution*
- s = +/- 0.1

$$x(t) = ae^{0.1t} + be^{-0.1t}$$

Plugging in the endpoints

$$x(0) = 5 = a + b$$

$$x(8) = 4 = 2.2255a + 0.4493b$$

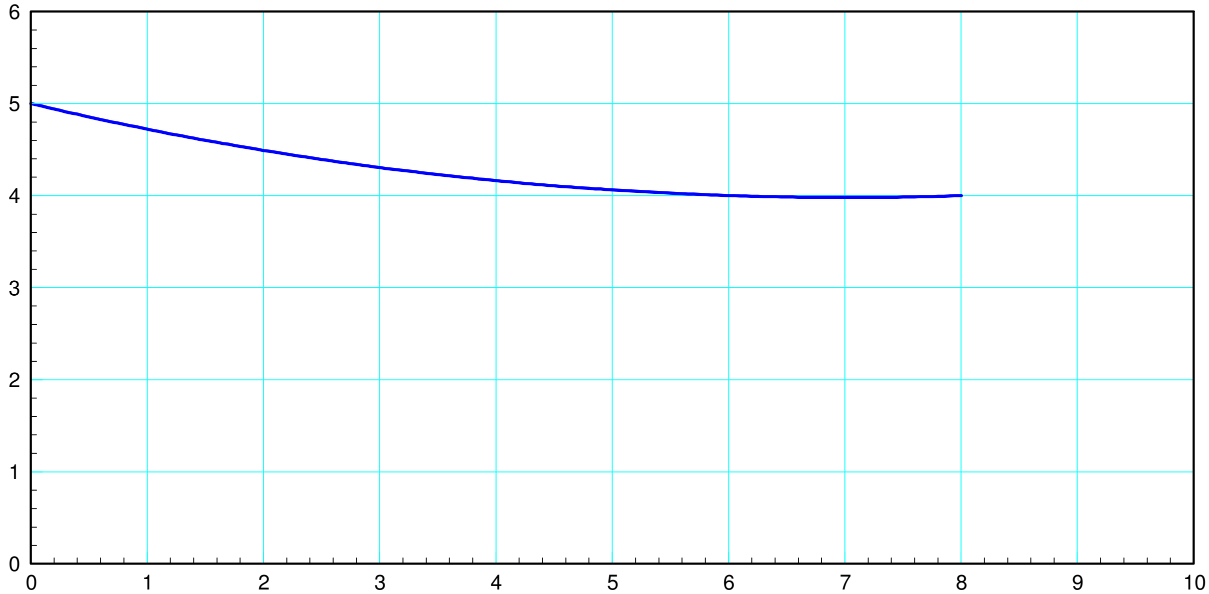
solving

a = 0.9871

b = 4.0129

and

$$x(t) = 0.9871e^{0.1t} + 4.0129e^{-0.1t}$$



Optimal path from (0,5) to (8,4)

5) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^8 (x^2 + 25u^2)dt$$

$$\dot{x} = -0.2x + u$$

x(0) = 5

x(8) = 4

The funcitonal is

$$F = x^2 + 25u^2 + m(\dot{x} + 0.2x - u)$$

Solving three sets of Euler LeGrange equations

$$\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\tfrac{d}{dt}(m) - (2x + 0.2m) = 0$$

$$\dot{m} = 2x + 0.2m$$

$$\frac{d}{dt}(F_{\dot{u}}) - F_u = 0$$

$$\tfrac{d}{dt}(0) - (50u - m) = 0$$

$$m = 50u$$

$$\frac{d}{dt}(F_{\dot{m}}) - F_m = 0$$

$$\tfrac{d}{dt}(0) - (\dot{x} + 0.2x - u) = 0$$

$$u = \dot{x} + 0.2x$$

Solving

$$\dot{m} = 50\dot{u}$$

$$\dot{m} = 2x + 0.2m$$

$$50\dot{u} = 2x + 0.2(50u)$$

$$50(\ddot{x} + 0.2\dot{x}) = 2x + 10(\dot{x} + 0.2x)$$

$$50\ddot{x} - 4x = 0$$

$$(50s^2 - 4)X = 0$$

$$s = \pm 0.2828$$

so

$$x(t) = ae^{0.2828t} + be^{-0.2828t}$$

Plugging in the endpoints

$$x(0) = 5 = a + b$$

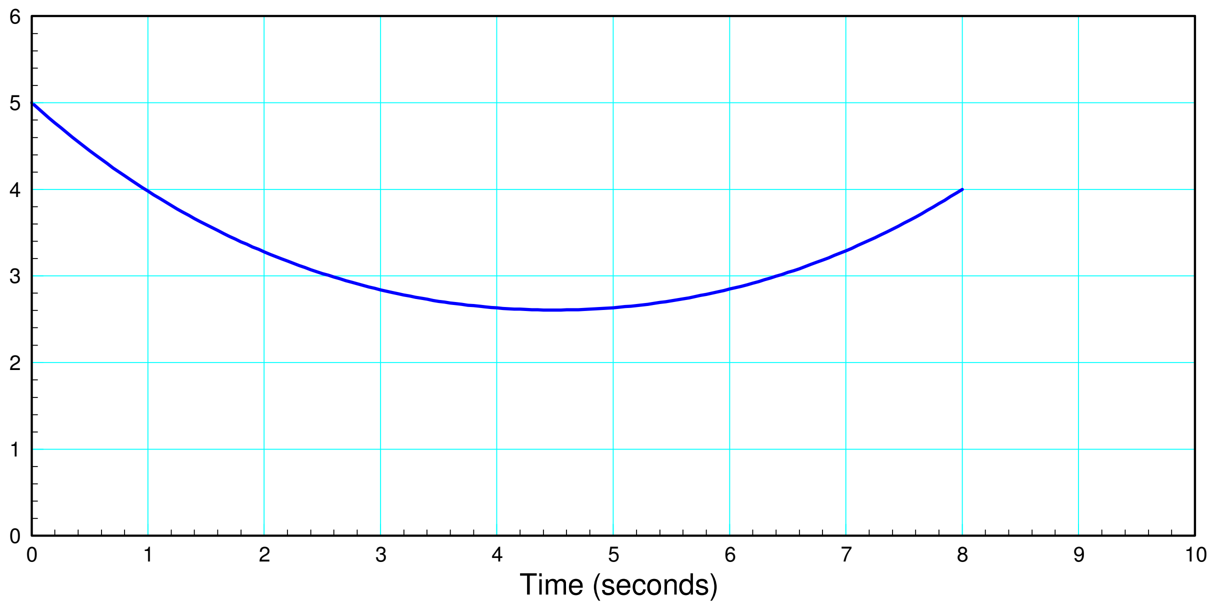
$$x(8) = 4 = 9.6061a + 0.1041b$$

a = 0.3662

b = 4.6338

and

$$x(t) = 0.3662e^{0.2828t} + 4.6338e^{-0.2828t}$$



Optimal path from (0,5) to (8,4)

LQG Control

6) Cart and Pendulum (HW #5): Design a full-state feedback control law of the form

$$U = K_r R - K_x X$$

for the cart and pendulum system from homework #5 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

Compare your results with homework #5

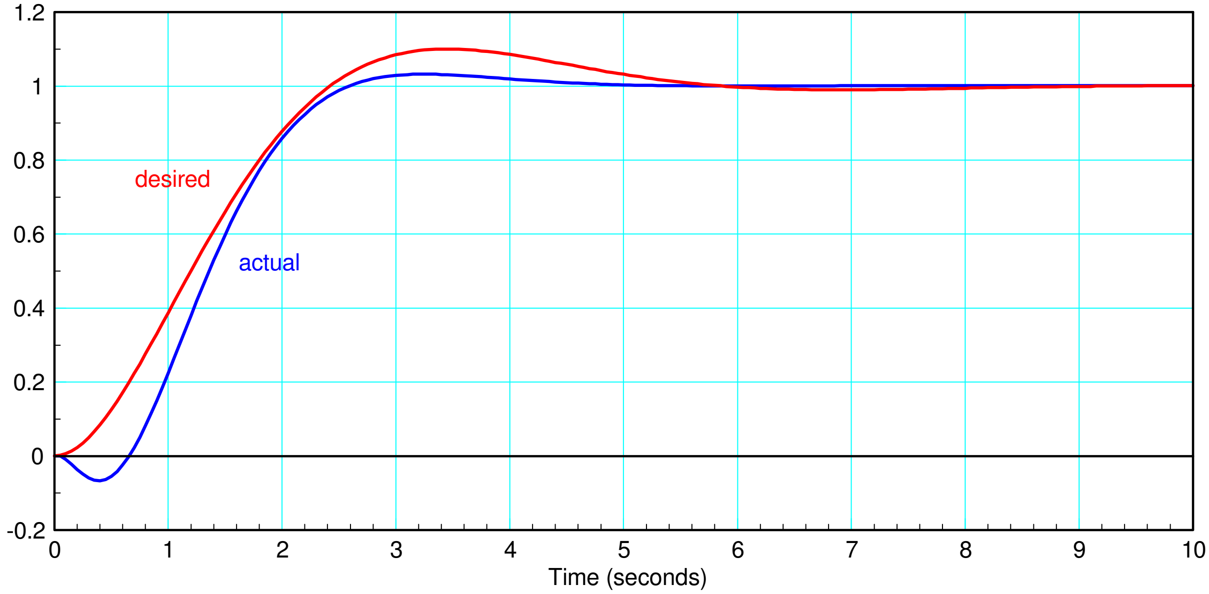
- Where are the closed-loop poles with pole placement and with LQG control?
- Are the feedback gains larger or smaller with LQG control?
- Which one works better?

In Matlab:

```
A = [0,0,1,0;0,0,0,1;0,-6.5333,0,0;0,16.333,0,0];
B = [0;0;0.333;-0.333];
C = [1,0,0,0];
t = [0:0.05:10]';

Gd = zpk([],[-0.666+j*0.91,-0.666-j*0.91],1.2717);
yd = step(Gd,t);

Qx = C'*C;
Qv = (C*A)'*(C*A);

Kx = lqr(A, B, 1*Qx + 0*Qv, 1);
DC = -C*inv(A-B*Kx)*B;
y = step3(A-B*Kx, B*Kr, C, 0, t, X0, 0*t+1);
plot(t,yd,'r',t,y,'b');
```

time passes...

```
Kx = lqr(A, B, 200*Qx + 0*Qv, 1);
DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
y = step3(A-B*Kx, B*Kr, C, 0, t, X0, 0*t+1);
plot(t,yd,'r',t,y,'b');

Kx =  -14.1421 -181.0162  -19.5164  -51.1747

Kr =  -14.1421

eig(A-B*Kx)

  -4.0070 + 0.2326i
  -4.0070 - 0.2326i
  -1.2642 + 1.1255i
  -1.2642 - 1.1255i
```


1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
1
2
3
4
5
6
7
8
9
10
Time (seconds)
desired
actual

7) Ball and Beam (HW #5): Design a full-state feedback control law of the form

$$U = K_r R - K_x X$$

for the ball and beam system from homework #6 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

```
A = [0,0,1,0;0,0,0,1;0,-7,0,0;-7.84,0,0,0];
B = [0;0;0;0.4];
C = [1,0,0,0];
```

Guess Q until the step response looks good

```
Kx = lqr(A, B, diag([10,10,10,1000]), 1);
DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
y = step3(A-B*Kx, B*Kr, C, 0, t, X0, 0*t+1);
plot(t,yd,'r',t,y,'b');

Kx =  -39.4535  112.9368  -25.5073   39.5561

Kr =  -19.8535

eig(A-B*Kx)

 -12.6758
  -0.7939 + 1.4775i
  -0.7939 - 1.4775i
  -1.5589

>>
```