ECE 463/663 - Test #3: Name ______________________

Calculus of Variations. Optimal Control. Spring 2020

Open Book, Open Notes. Calculators, Matlab, Internet all permitted - just not other people.

Please sign if possible (i.e. you worked alone):

No Aid Given Received Observerd: ______________________

Calculus of Variations: Fixed endpoints

1) Find the function which minimizes the following funcitonal:

$$J = \int_0^8 \left(2x^2 + 3\dot{x}^2\right) dt$$

subject to the constraints

$x(0) = 6$

$x(8) = 4$

$$F = 2x^2 + 3\dot{x}^2$$

$$\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\frac{d}{dt}(6\dot{x}) - (4x) = 0$$

$$6\ddot{x} - 4x = 0$$

$$(6s^2 - 4)X = 0$$

$$s = \pm 0.8165$$

$$x(t) = ae^{0.8165t} + be^{-0.8165t}$$

Plug in the endpoints

$$x(0) = 6 = a + b$$

$$x(8) = 4 = 686.7516a + 0.0015b$$

```
A = [1,1 ; exp(8*0.8165),exp(-8*0.8165)]
B = [6;4]
C = inv(A)*B

     0.0058
     5.9942
```

$$x(t) = 0.0058e^{0.8165t} + 5.9942e^{-0.8165t}$$

2) Calculus of Variations: Free Endpoint. Find the function which minimizes the following funcitonal:

$$J = \int_0^2 \left(2x^2 + 3\dot{x}^2 \right) dt$$

subject to the constraints

x(0) = 6

x(8) = free

From problem #1

$$F = 2x^2 + 3\dot{x}^2$$

$$s = \pm 0.8165$$

$$x(t) = ae^{0.8165t} + be^{-0.8165t}$$

Plugging in the endpoints

$$x(0) = 6$$

$$F_{\dot{x}}(8) = 6\dot{x}(8) = 0$$

$$\dot{x} = 0.8165ae^{0.8165t} - 0.8165be^{-0.8165t}$$

$$560.748a - 0.0012b = 0$$

```
A = [1,1 ; 0.8165*exp(8*0.8165),-0.8165*exp(-8*0.8165)]
B = [6;0]
C = inv(A)*B

    0.000012721174735
    5.999987278825266
```

$$x(t) = 0.0000127e^{0.8165t} + 5.9999872e^{-0.8165t}$$

3) Calculus of Variations: You are to build a road from point a) to point b). The cost per unit distance of the road is proportional to the distance from origin

$$J = \int_a^b \sqrt{x^2 + y^2}\,\sqrt{1 + \dot{y}^2} \cdot dx$$

Determine the differential equations that the solution must satisfy (don't solve - it's kind of nasty).

$$F = \sqrt{x^2 + y^2}\,\sqrt{1 + \dot{y}^2}$$

$$\frac{d}{dx}\left(F_{\dot{y}}\right) - F_y = 0$$

$$\frac{d}{dx}\left(\frac{\dot{y}\sqrt{x^2+y^2}}{\sqrt{1+\dot{y}^2}}\right) - \left(\frac{y\sqrt{1+\dot{y}^2}}{\sqrt{x^2+y^2}}\right) = 0$$

$$\left(\frac{\ddot{y}\sqrt{x^2+y^2}}{\sqrt{1+\dot{y}^2}}\right) + \left(\frac{\dot{y}(x+y\dot{y})}{\sqrt{x^2+y^2}\,\sqrt{1+\dot{y}^2}}\right) - \left(\frac{\dot{y}^2\ddot{y}\sqrt{x^2+y^2}}{\left(1+\dot{y}^2\right)^{3/2}}\right) - \left(\frac{y\sqrt{1+\dot{y}^2}}{\sqrt{x^2+y^2}}\right) = 0$$

4) Optimal Control: Given the following cost function and constraing:

$$J = \int \left(x^2 + 4u^2 \right) dt \qquad \dot{x} = -2x + 3u$$

$$F = x^2 + 4u^2 + m(-2x + 3u - \dot{x})$$

Determine the three differential equations that the optimal solution must satisfy:

(1) $\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$

$$\frac{d}{dt}(-m) - (2x - 2m) = 0$$

$$-\dot{m} - 2x + 2m = 0$$

(2) $\frac{d}{dt}(F_{\dot{u}}) - F_u = 0$

$$(0) - (8u + 3m) = 0$$

(3) $\frac{d}{dt}(F_{\dot{m}}) - F_m = 0$

$$(0) - (-2x + 3u - \dot{x}) = ($$

Determine the optimal path for x(t), subject to

x(0) = 6

x(8) = 4

$$m = -\tfrac{8}{3}u$$

$$2x - 2m + \dot{m} = 0$$

$$2x + \tfrac{16}{3}u - \tfrac{8}{3}\dot{u} = 0$$

$$3x + 8u - 4\dot{u} = 0$$

$$3x = (4s - 8)U$$

$$U = \left(\tfrac{3}{4s-8} \right) X$$

$$-2x + 3u - \dot{x} = 0$$

$$(-2 - s)X + 3\left(\left(\frac{3}{4s-8}\right)X\right) = 0$$

$$(4s - 8)(-2 - s)X + 9X = 0$$

$$(-4s^2 - 8s + 8s + 16 + 9)X = 0$$

$$(-4s^2 + 25)X = 0$$

$$s = \pm\sqrt{\frac{25}{4}} = \pm 2.500$$

$$x(t) = a\ e^{-2.5t} + b\ e^{2.5t}$$

Plug in the endpoints

$$x(0) = 6 = a + b$$

$$x(8) = 4 = a\ e^{-16} + b\ e^{16}$$

a = 5.999999991755385

b = 0.000000008244614

$$x(t) \approx 6\ e^{-2.5t}$$

5) Find a feedback control law so that the following system:

$$sX = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

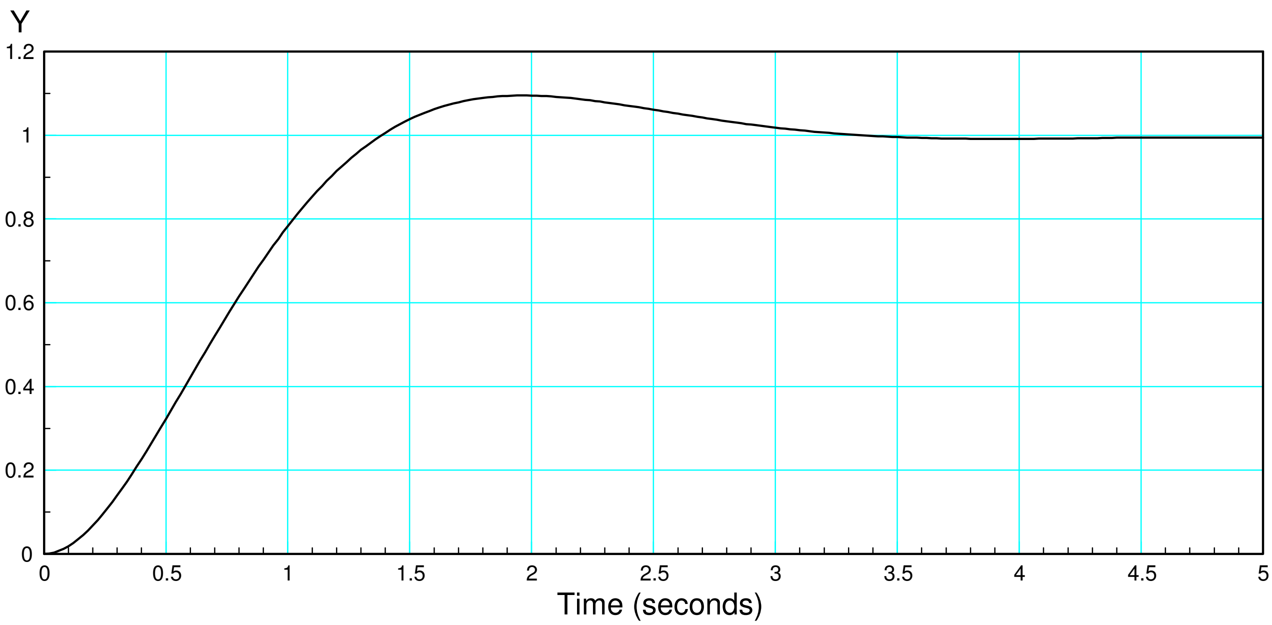
$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

has the following step response: (your pick which method you use)

Give

- The method you used (pole placement, LQR, LQG/LTR)
- Your resulting cotrol law, and
- The step response of your closed-loop system

Note: matlab is encouraged for this problem.



The settling time is 4 seconds

The overshoot is 10%

The damping ratio is 0.5912

The angle is 53.76 degrees

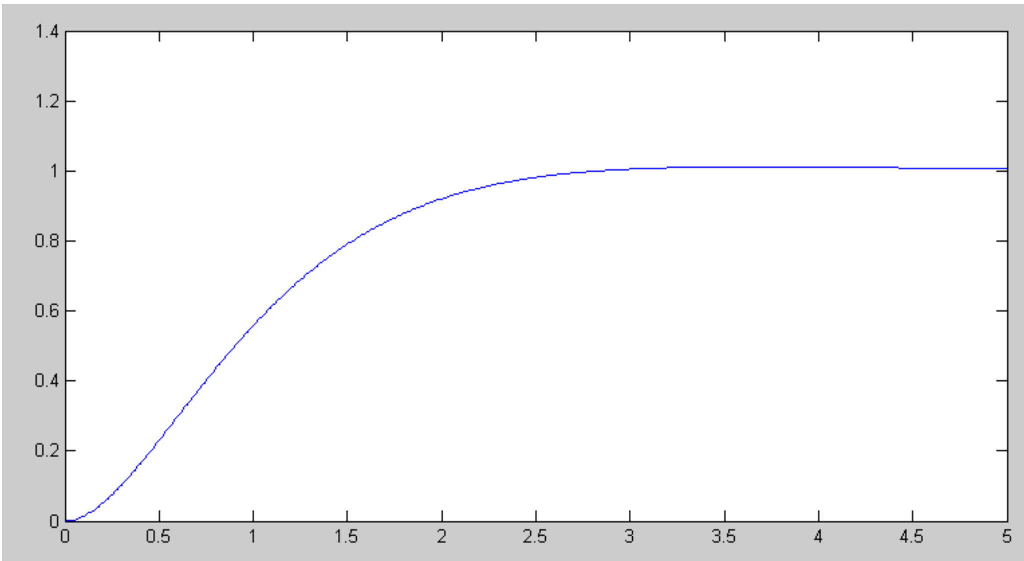
$$s = -1 \pm j1.36$$

```
A4 = [A, 0*B ; C, 0]

    -2     1     0     0
     1    -2     1     0
     0     1    -1     0
     1     0     0     0

B4 = [B; 0];
C4 = [C, 0];
K4 = lqr(A4, B4, diag([0,0,0,10]),1)
```

Kx Kz

K4 = 1.3684 0.5109 0.2093 3.1623

```
G4 = ss(A4-B4*K4, B4r,C4,0);
y = step(G4,t);
plot(t,y)
```



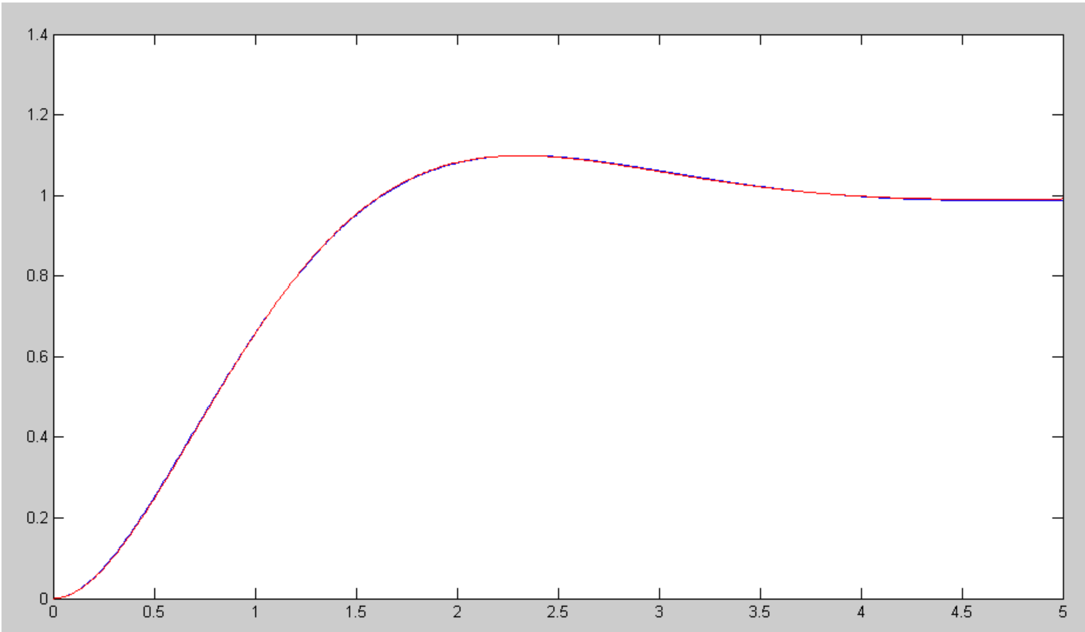
Solution #2: Let fminsearch find Q:

```
function [ J ] = Prob5( Z )

Q = diag(Z);
R = 1;
A = [-2,1,0;1,-2,1;0,1,-1];
B = [1;0;0];
C = [1,0,0];

A4 = [A, 0*B ; C, 0];
B4 = [B ; 0];
B4r = [0*B ; -1];
C4 = [C, 0];

K4 = lqr(A4, B4, Q, R);

t = [0:0.01:5]';
G4 = ss(A4-B4*K4, B4r, C4, 0);
y = step(G4, t);

Gd = tf(2.84,[1,2,2.84]);
Yd = step(Gd, t);

E = Yd - y;
J = sum(E.^2);

plot(t,y,'b',t,Yd,'r');
pause(0.01);

end
```

Result:

```
[Z,e] = fminsearch('Prob5',[0,0,0,1])

Z =   -5.4824   -2.6185    6.1926    8.9260
e =    0.0022
```

K4 =   0.2962   0.3898   0.4378   2.9876



Desired Reponse (red) and Actual Response (blue)