ECE 463/663 - Test #3: Name ______________________

Calculus of Variations. Optimal Control. Spring 2021

Open Book, Open Notes. Calculators, Matlab permitted - just not other people.

Calculus of Variations: Fixed endpoints

1) Find the function which minimizes the following funcitonal:

$$J = \int_0^5 \left(2x^2 + 10\dot{x}^2\right) dt$$

subject to the constraints

x(0) = 6

x(5) = 4

Solve the Euler LaGrange equation

$$\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\frac{d}{dt}(20\dot{x}) - (4x) = 0$$

$$20\ddot{x} - 4x = 0$$

using LaPlace notation

$$(20s^2 - 4)X = 0$$

$$s = \pm\sqrt{\frac{1}{5}} = \pm 0.4472$$

meaning

$$x(t) = ae^{0.4472t} + be^{-0.4472t}$$

Plugging in the endpoints

$$x(0) = 6 = a + b$$

$$x(5) = 4 = 9.3558a + 0.1069b$$

solving

```
>> A = [1,1 ; exp(0.4472*5),exp(-0.4472*5)]

A =

    1.0000    1.0000
    9.3558    0.1069

>> ab = inv(A) * [6;4]

a    0.3631
b    5.6369
```

$$x(t) = 0.3631e^{0.4472t} + 5.6369e^{-0.4472t}$$

2) Calculus of Variations: Free Endpoint.

Find the function which minimizes the following funcitonal:

$$J = \int_0^5 \left(2x^2 + 10\dot{x}^2\right) dt$$

subject to the constraints

x(0) = 6

x(5) = free

This will have the same solution as problem #1

$$x(t) = ae^{0.4472t} + be^{-0.4472t}$$

Left endpoint:

$$x(0) = 6 = a + b$$

Right endpoint:

$$F_{\dot{x}} = 0$$

$$\dot{x} = 0$$

$$\dot{x}(t) = 0.4472ae^{0.4472t} - 0.4472be^{-0.4472t}$$

$$\dot{x}(5) = 0 = 4.1839a - 0.0478b$$

Solving

```
>> A = [1,1 ; 0.4472*exp(5*0.4472),-0.4472*exp(-0.4472*5)]

    1.0000     1.0000
    4.1839    -0.0478

>> ab = inv(A)*[6;0]

a     0.0678
b     5.9322
```

$$x(t) = 0.0678e^{0.4472t} + 5.9322e^{-0.4472t}$$

3) Optimal Control: Given the following cost function:

$$J = \int \left(2x^2 + 10u^2 \right) dt$$

subject to the constraint

- $\dot{x} = 0.2x + u$ (unstable)
- $x(0) = 6$
- $x(5) = 4$

Solution: First, set up the functional:

$$F = x^2 + 5u^2 + m(\dot{x} - 0.2x - u)$$

Determine the three differential equations that the optimal solution must satisfy:

(1) $$\frac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\frac{d}{dt}(m) - (2x - 0.2m) = 0$$

$$\dot{m} - 2x + 0.2m = 0$$

(2) $$\frac{d}{dt}(F_{\dot{u}}) - F_u = 0$$

$$\frac{d}{dt}(0) - (10u - m) = 0$$

$$m = 10u$$

(3) $$\frac{d}{dt}(F_{\dot{m}}) - F_m = 0$$

$$\frac{d}{dt}(0) - (\dot{x} - 0.2x - u) = 0$$

$$u = \dot{x} - 0.2x$$

Eliminate m: substitute (2) in to (1)

$$10\dot{u} - 2x + 2u = 0$$

Eliminate u: substitute (3) for u

$$10(\ddot{x} - 0.2\dot{x}) - 2x + 2(\dot{x} - 0.2x) = 0$$

$$10\ddot{x} - 2.4x = 0$$

using LaPlace notation

$$\left(s^2 - 0.24\right)X = 0$$

$$s = \pm\sqrt{0.24}$$

meaning

$$x(t) = ae^{0.4899t} + be^{-0.4899t}$$

Determine the optimal path for x(t), subject to

x(0) = 6

x(5) = 4

Plugging in the endpoints

$$x(t) = 6 = a + b$$

$$x(5) = 4 = 11.5826a + 0.0863b$$

Solving

```
>> A = [1,1 ; exp(0.4899*5),exp(-0.4899*5)]

    1.0000    1.0000
   11.5826    0.0863

>> ab = inv(A) * [6;4]

a    0.3029
b    5.6971

>>
```

giving

$$x(t) = 0.3029 \cdot e^{0.4899t} + 5.6971 \cdot e^{-0.4899t}$$

4) Optimal Control: Non-Quadratic Cost Function.

Find the optimal path, x(t), to minize the following cost function

$$J = \int \left(x^4 + 5u^2 \right) dt$$

subject to

$$\dot{x} = 0.2x + u$$

Solution: The functional becomes

$$F = x^4 + 5u^2 + m(\dot{x} - 0.2x - u)$$

Solving the three Euler LaGrange equations:

(1) $$\tfrac{d}{dt}(F_{\dot{x}}) - F_x = 0$$

$$\tfrac{d}{dt}(m) - \left(3x^3 - 0.2m \right) = 0$$

$$\dot{m} - 3x^3 + 0.2m = 0$$

(2) $$\tfrac{d}{dt}(F_{\dot{u}}) - F_u = 0$$

$$\tfrac{d}{dt}(0) - (10u - m) = 0$$

$$m = 10u$$

(3) $$\tfrac{d}{dt}(F_{\dot{m}}) - F_m = 0$$

$$\tfrac{d}{dt}(0) - (\dot{x} - 0.2x - u) = 0$$

$$u = \dot{x} - 0.2x$$

Substitute (2) into (1)

$$10\dot{u} - 3x^3 + 2u = 0$$

Substitute (3) for u

$$10(\ddot{x} - 0.2\dot{x}) - 3x^3 + 2(\dot{x} - 0.2x) = 0$$

$$10\ddot{x} - 3x^3 - 0.4x = 0$$

$$\ddot{x} = 0.3x^3 + 0.4x$$

This is a nonlienar differential equation. LaPlace notation doesn't help here.

Bonus: Find x(t) for problem #4.

We know x(0) = 6. We don't know $\dot{x}(0)$.

If you knew x'(0), you could solve for x(t) using numerical integration. The correct initial condition is whatever causes the path to pass through the endpoint

x(5) = 4

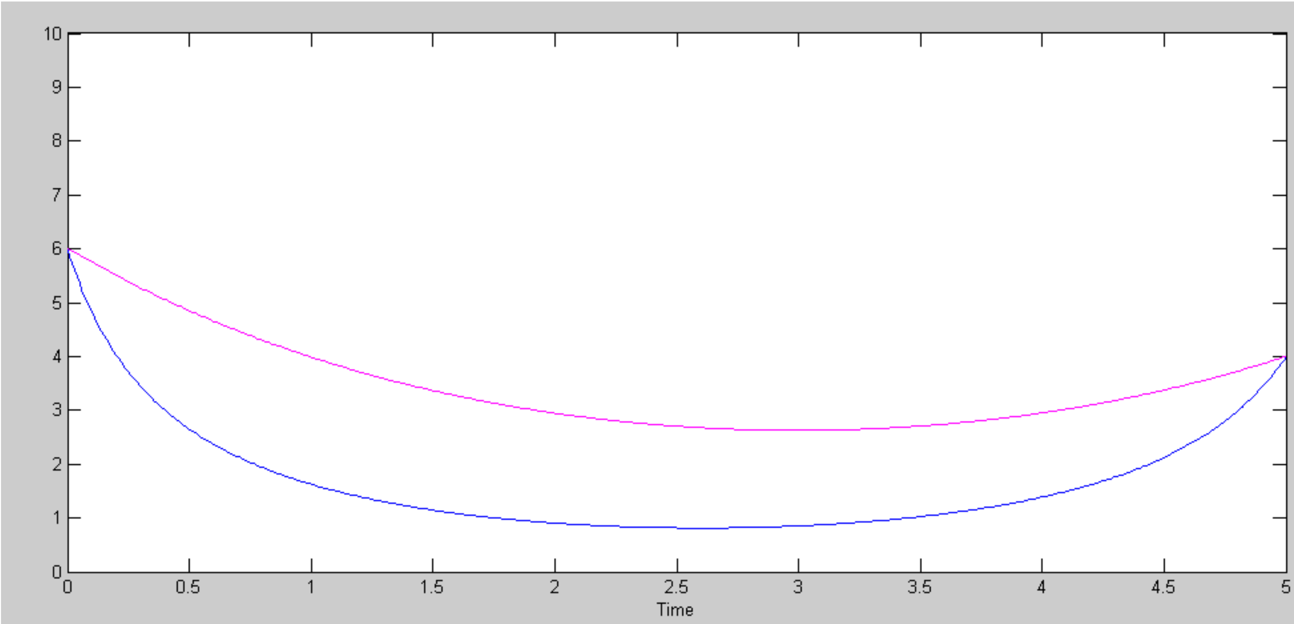
Create a cost function in matlab which

- Is passed dx(0)
- Integrates for 5 seconds, then
- Returns the error from x(5) = 4

```
function [ J ] = Bonus( z )

dt = 0.01;
t = [0:dt:5]';
x = 0*t;
n = length(t);

x(1) = 6;
dx = z;

for i=2:n
    ddx = 0.3*(x(i-1))^3 + 0.4*x(i-1);

    dx = dx + ddx*dt;
    x(i) = x(i-1) + dx*dt;
    x(i) = min(x(i), 100);
    x(i) = max(x(i), -100);
end

plot(t,x);
ylim([0,10]);
xlim([0,5]);

pause(0.01);
J = (x(n) - 4)^2;

end
```

Optimize using fminsearch

```
>> [Z,e] = fminsearch('Bonus',-14.72)

Z =  -14.7797

e =  9.2588e-006
```



Optimal Path for x(t): Problem #3 (magenta) and Problem #4 (blue)