ECE 463/663 - Homework #3

Canonical Forms, Similarity Transforms, LaGrangian Dynamics, Controllability. Due Monday, Feb 1st

Please make the subject "ECE 463 HW#3" if submitting homework electronically to Jacob_Glower@yahoo.com (or on blackboard)

Problem 1-3) For the system

$$Y = \left(\frac{20(s+6)}{(s+1)(s+7)(s+10)}\right)X$$

1) Express this system in controller canonical form. (Give the A, B, C, D matrices)

Multiply this out

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>> poly([-1,-7,-10])

ans =      1    18    87    70
```

$$Y = \left(\frac{20s+120}{s^3+18s^2+87s+70}\right)X$$

By inspection, controller form is then

$$sX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -70 & -87 & -18 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 120 & 20 & 0 \end{bmatrix} X$$

2) Express this system in cascade form

Rewrite this as

$$Y = \left(\frac{a}{s+1}\right) + \left(\frac{b}{(s+1)(s+7)}\right) + \left(\frac{c}{(s+1)(s+7)(s+10)}\right)$$

Putting this over a common denominator, the numerator becomes

$$a(s+7)(s+10) + b(s+10) + c = 20s + 120$$

a = 0

b = 20

c = -80

$$sX = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -7 & 0 \\ 0 & 1 & -10 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 20 & -80 \end{bmatrix} X$$

3) Express this system in Jordan (diagonal) form

Use partial fractions

$$\left(\frac{20(s+6)}{(s+1)(s+7)(s+10)}\right) = \left(\frac{1.852}{s+1}\right) + \left(\frac{1.111}{s+7}\right) + \left(\frac{-2.963}{s+10}\right)$$

In Jordan form then

$$sX = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -10 \end{bmatrix} X + \begin{bmatrix} 1.852 \\ 1.111 \\ -2.963 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$

4) Assume a system's dynamics are

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 1 & -10 & 0 \\ 0 & 1 & -15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} V_0$$

$$Y = V_3$$

Express these dynamic with the change in variable

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} V_3 \\ V_2 - 2V_3 \\ V_1 - 5V_2 + 5V_3 \end{bmatrix}$$

In matrix form

$$Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -5 & 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = T^{-1} X$$

```
>> A = [-5,0,0 ; 1,-10,0 ; 0,1,-15];
>> B = [1;2;3];
>> C = [0,0,1];
>> Ti = [0,0,1 ; 0,1,-2 ; 1,-5,5];
>> T = inv(Ti);
>> Az = inv(T)*A*T

   -13      1      0
    11     -7      1
   -15      5    -10

>> Bz = inv(T)*B

     3
    -4
     6

>> Cz = C*T

Cz =     1      0      0
```

meaning

$$sZ = \begin{bmatrix} -13 & 1 & 0 \\ 11 & -7 & 1 \\ -15 & 5 & -10 \end{bmatrix} Z + \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} Z$$

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>>
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LaGrangian Dynamics

A 1kg ball is rolling in a bowl with the shape

$$y = 1 - \cos\left(\frac{x}{2}\right)$$

6) Determine the kinetic and potential energy of this ball as a function of x: Gravity is in the -y direction.

$$PE = mgy = 9.8\left(1 - \cos\left(\frac{x}{2}\right)\right)$$

Assuming a solid sphere

$$KE = 0.7m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{y} = \frac{1}{2}\sin\left(\frac{x}{2}\right)\dot{x}$$

Substituting

$$KE = 0.7\left(\dot{x}^2 + \left(\frac{1}{2}\sin\left(\frac{x}{2}\right)\dot{x}\right)^2\right)$$

$$KE = 0.7\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right)\dot{x}^2$$

7) Determine the dynamics for this ball as it rolls in the bowl

$$L = KE - PE$$

$$L = 0.7\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right)\dot{x}^2 - 9.8\left(1 - \cos\left(\frac{x}{2}\right)\right)$$

The dynamics are

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(1.4\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right)\dot{x}\right) - \left(\frac{0.7}{4}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\dot{x}^2 - \frac{9.8}{2}\sin\left(\frac{x}{2}\right)\right)$$

$$F = 1.4\ddot{x}\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right) + \left(\frac{1.4}{4}\right)\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\dot{x}^2$$

$$-\frac{0.7}{4}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\dot{x}^2 + \frac{9.8}{2}\sin\left(\frac{x}{2}\right)$$

$$F = 1.4\ddot{x}\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right) + \left(\frac{0.7}{4}\right)\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\dot{x}^2 + \frac{9.8}{2}\sin\left(\frac{x}{2}\right)$$

With F = 0

$$\ddot{x} = -\frac{\left(\frac{0.7}{4}\right)\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\dot{x}^2 + \frac{9.8}{2}\sin\left(\frac{x}{2}\right)}{1.4\left(1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)\right)}$$

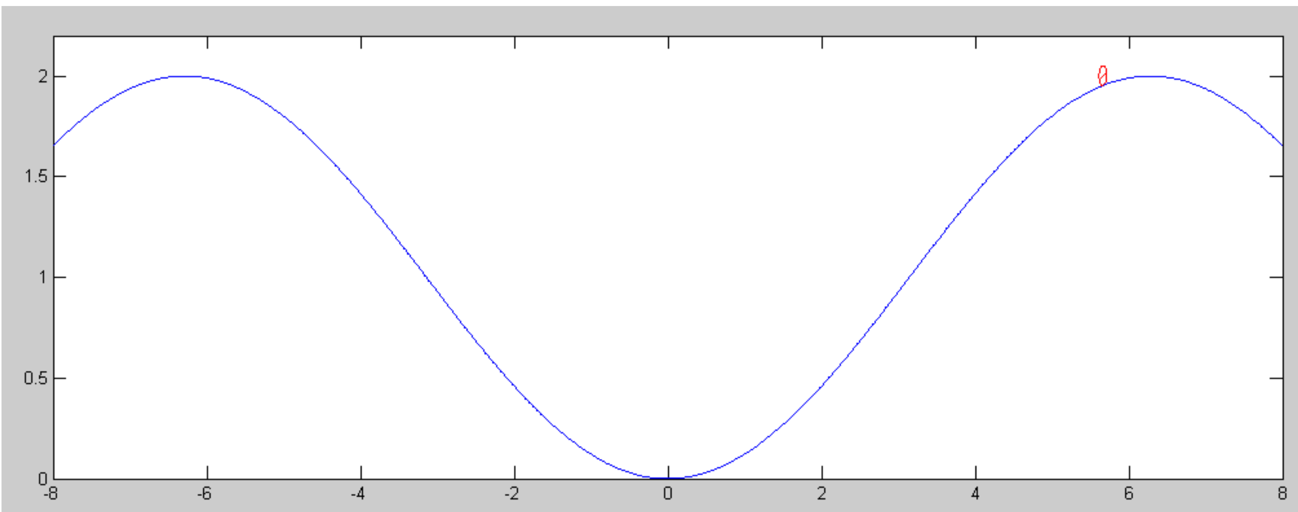
Note for animation:

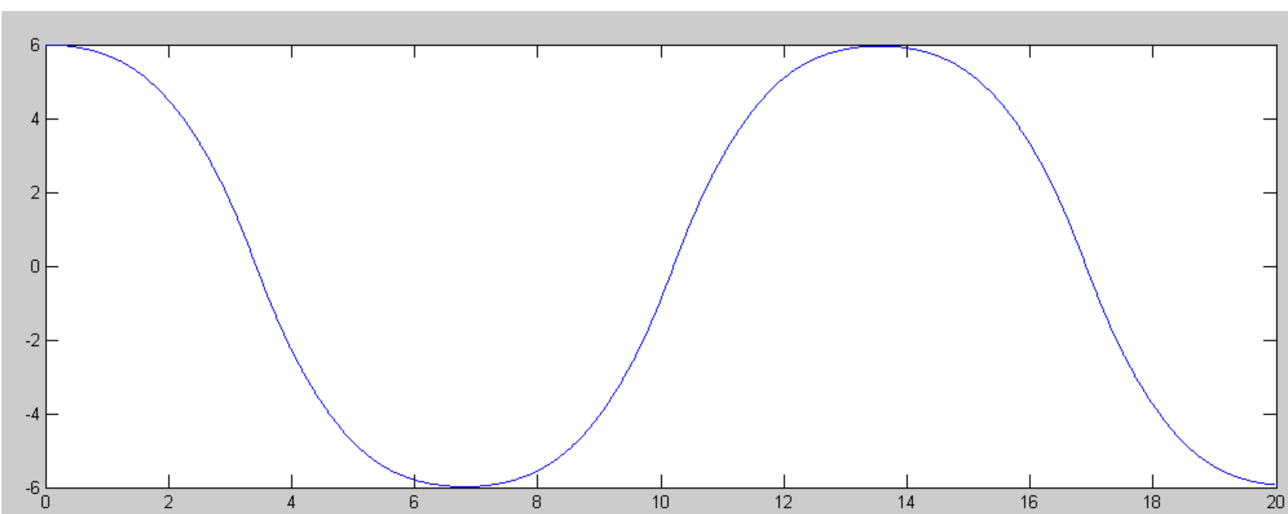
$$d = r\theta$$

$$d\theta = \tfrac{1}{r}\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$d\theta = \tfrac{1}{r}\left(\sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\right)dx$$

$$d\theta = \tfrac{1}{r}\left(\sqrt{1 + \left(\tfrac{1}{2}\sin\left(\tfrac{x}{2}\right)\dot{x}\right)^2}\right)dx$$





x vs time

Code

```
% Ball in a bowl.  ECE 463 Spring 2021

x = 6;
dx = 0;
dt = 0.01;
t = 0;
g = 9.8;
L = 0;
q = 0;  % angle of ball for animation
r = 0.05; % radius of ball for animation

Yd = []; % display matrix

while(t < 20)

% compute the acceleration

ddx = -( 0.7/4*sin(x/2)*cos(x/2)*dx*dx + g/2*sin(x/2) ) / ( 1.4*(1 + (sin(x/2))^2 )
);
dq = sqrt(1 + (0.5*sin(x/2)*dx)^2) /r * dx;

% integrate

dx = dx + ddx*dt;
x = x + dx*dt;
q = q + dq*dt;
t = t + dt;

% display the ball
 y = 1 - cos(x/2);

 x1 = [-8:0.01:8]';
 y1 = 1 - cos(x1/2);

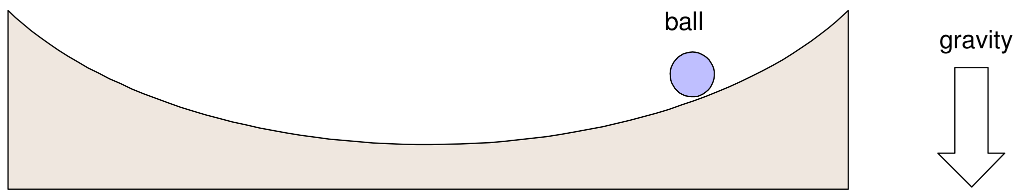
 % draw the ball
 i = [0:0.01:1]' * 2 * pi;
 xb = r*cos(i) + x;
 yb = r*sin(i) + y + r;

 % line through the ball
 Q = [0, pi] - q;
 xb1 = 0.05*cos(Q) + x;
 yb1 = 0.05*sin(Q) + y + r;

 plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');
 ylim([0,2.2]);
 xlim([-8,8]);
 pause(0.01);
 Yd = [Yd ; x];

end

t = [0:length(Yd)-1]' * dt;
plot(t,Yd);
```


ball
gravity