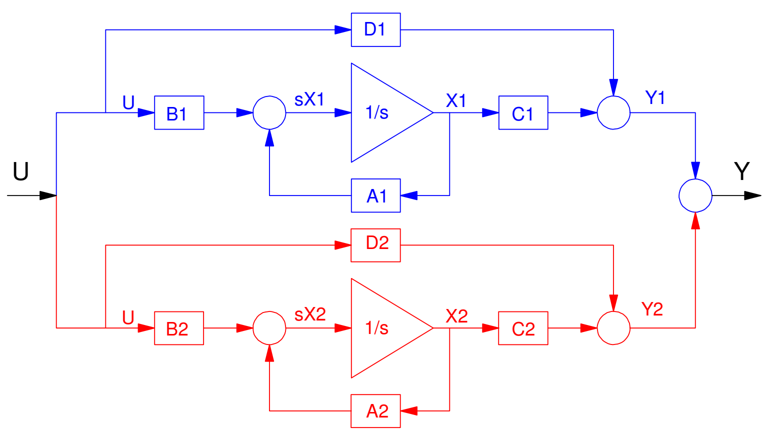
ECE 463/663 - Homework #4

Block Diagrams and LaGrangian Dynamics. Due Monday, February 10th

1) Determine the state-space model for two systems in parallel:



Write the differential equations

$$sX_1 = A_1 X_1 + B_1 U$$

$$sX_2 = A_2 X_2 + B_2 U$$

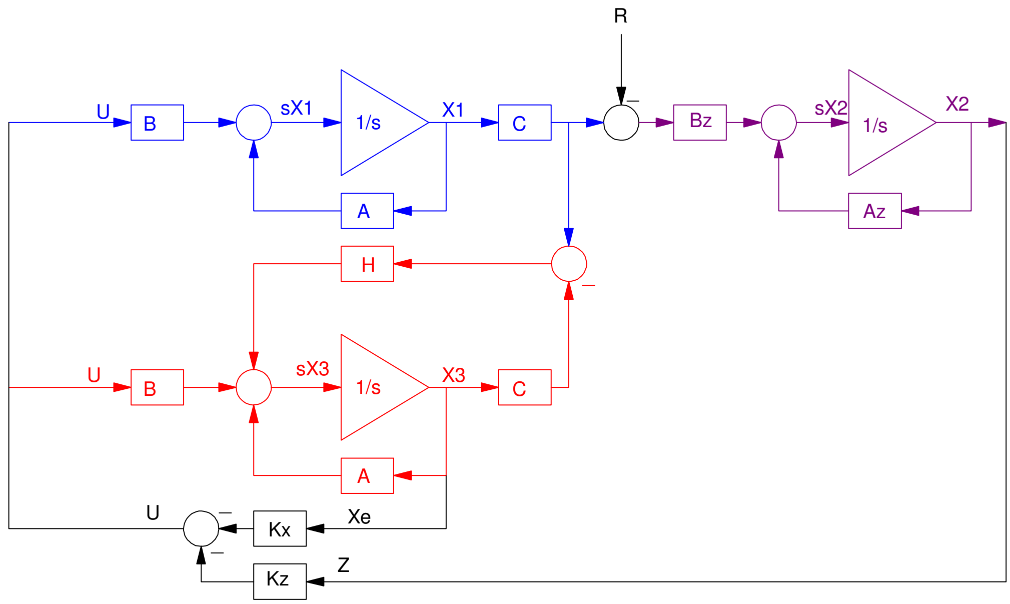
$$Y = C_1 X_1 + C_2 X_2 + D_1 U + D_2 U$$

Place in matrix form

$$s\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U$$

$$Y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [D_1 + D_2]U$$

2) Determine the state-space model for the following system:



write the differential equations

$$sX_1 = AX_1 - BK_xX_3 - BK_zX_2$$

$$sX_2 = A_zX_2 + B_zCX_1 - B_zR$$

$$sX_3 = AX_3 - HCX_3 + HCX_1 - BK_xX_3 - BK_zX_2$$

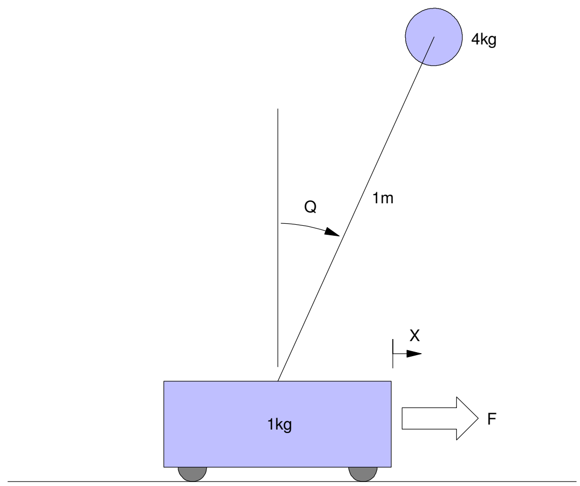
Place in matrix form

$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ B_zC & A & 0 \\ HC & -BK_z & A - HC - BK_x \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -B_z \\ 0 \end{bmatrix}R$$

3) (30pt) Derive the dynamics for an inverted pendulum where

- m1 = 4kg (mass of ball)
- m2 = 1kg (mass of cart)
- L = 1.0m (length of arm)

Fine the linearized dynamics at x = 0, θ = 0



Mass #1

$$x_1 = x \qquad y_1 = 0$$

$$\dot{x}_1 = \dot{x} \qquad \dot{y}_1 = 0$$

$$PE = 0$$

$$KE = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\dot{x}^2$$

Mass #2

$$x_2 = x + \sin\theta \qquad y_2 = \cos\theta$$

$$\dot{x}_2 = \dot{x} + \cos\theta\dot{\theta} \qquad \dot{y}_2 = -\sin\theta\dot{\theta}$$

$$PE = 4g\cos\theta$$

$$KE = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$KE = 2\left(\left(\dot{x} + \cos\theta\dot{\theta}\right)^2 + \left(-\sin\theta\dot{\theta}\right)^2\right)$$

$$KE = 2\dot{x}^2 + 2\dot{\theta}^2 + 4\cos\theta\dot{x}\dot{\theta}$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left(2.5\dot{x}^2 + 2\dot{\theta}^2 + 4\cos\theta\dot{x}\dot{\theta}\right) - (4g\cos\theta)$$

To find the dynamics, use the Euler LaGrange equation

$$L = \left(2.5\dot{x}^2 + 2\dot{\theta}^2 + 4\cos\theta\dot{x}\dot{\theta}\right) - (4g\cos\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}\left(5\dot{x} + 4\cos\theta\dot{\theta}\right) - (0)$$

$$F = 5\ddot{x} + 4\cos\theta\ddot{\theta} - 4\sin\theta\dot{\theta}^2$$

$$L = \left(4\dot{x}^2 + 2\dot{\theta}^2 + 4\cos\theta\dot{x}\dot{\theta}\right) - (4g\cos\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \tfrac{d}{dt}\left(4\dot{\theta} + 4\cos\theta\dot{x}\right) - \left(-4\sin\theta\dot{x}\dot{\theta} + 4g\sin\theta\right)$$

$$T = 4\ddot{\theta} + 4\cos\theta\ddot{x} - 4\sin\theta\dot{x}\dot{\theta} + 4\sin\theta\dot{x}\dot{\theta} - 4g\sin\theta$$

$$T = 4\ddot{\theta} + 4\cos\theta\ddot{x} - 4g\sin\theta$$

So, the dynamics are

$$F = 5\ddot{x} + 4\cos\theta\ddot{\theta} - 4\sin\theta\dot{\theta}^2$$

$$T = 4\ddot{\theta} + 4\cos\theta\ddot{x} - 4g\sin\theta$$

In Matrix form

$$\begin{bmatrix} 5 & 4\cos\theta \\ 4\cos\theta & 4 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ T \end{bmatrix} + \begin{bmatrix} 4\sin\theta\dot{\theta}^2 \\ 4g\sin\theta \end{bmatrix}$$

Linearizing about zero with T = 0

$$\begin{bmatrix} 5 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}F + \begin{bmatrix} 0 \\ 4g\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}F + \begin{bmatrix} -4g\theta \\ 5g\theta \end{bmatrix}$$

Putting this in state-space form

$$s\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4g & 0 & 0 \\ 0 & 5g & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}F$$

or

$$s\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -39.2 & 0 & 0 \\ 0 & 49.0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}F$$

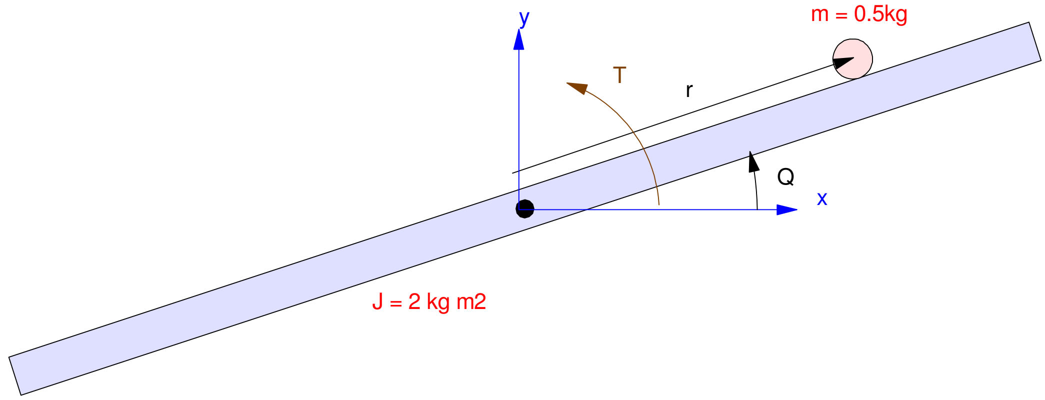
The poles are at

```
>> g = 9.8;
>> A = [0,0,1,0;0,0,0,1;0,-4*g,0,0;0,5*g,0,0]

A =

         0         0    1.0000         0
         0         0         0    1.0000
         0  -39.2000         0         0
         0   49.0000         0         0

>> B = [0;0;1;-1]

B =

     0
     0
     1
    -1

>> eig(A)

ans =

         0
         0
    7.0000
   -7.0000
```

4) (30pt) Derive the dynamics for a ball and beam system where

- J = 2.0 kg m2 (the inertia of the beam)
- m = 0.5kg (the mass of the ball)

Find the linearized dynamics at r = 1.0m, θ = 0



Position of the ball:

$$x_1 = r\cos\theta \qquad y_1 = r\sin\theta$$

$$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y}_1 = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

The potential and kinetic energy (assuming a solid sphere) are:

$$PE = mgy_1 = mgr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \dot{\theta}^2 + \tfrac{1}{2}m\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \left(1 + \tfrac{1}{2}mr^2\right)\dot{\theta}^2 + 0.7m\dot{r}^2$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left(\left(1 + \tfrac{1}{2}mr^2\right)\dot{\theta}^2 + 0.7m\dot{r}^2\right) - (mgr\sin\theta)$$

With m = 0.5kg

$$L = \left(\dot{\theta}^2 + 0.25r^2\dot{\theta}^2 + 0.35\dot{r}^2\right) - (0.5gr\sin\theta)$$

Force on the Ball

$$L = \left(\dot{\theta}^2 + 0.25r^2\dot{\theta}^2 + 0.35\dot{r}^2\right) - (0.5gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial r'}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(0.7\dot{r}) - \left(0.5r\dot{\theta}^2 - 0.5g\sin\theta\right)$$

$$F = 0.7\ddot{r} - 0.5r\dot{\theta}^2 + 0.5g\sin\theta$$

Torque on the Beam

$$L = \left(\dot{\theta}^2 + 0.25r^2\dot{\theta}^2 + 0.35\dot{r}^2\right) - (0.5gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(2\dot{\theta} + 0.5r^2\dot{\theta}\right) - (-0.5gr\cos\theta)$$

$$T = 2\ddot{\theta} + 0.5r^2\ddot{\theta} + r\dot{r}\dot{\theta} + 0.5gr\cos\theta$$

Putting it together

$$\begin{bmatrix} 0.7 & 0 \\ 0 & 2+0.5r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0.5r\dot{\theta}^2 - 0.5g\sin\theta \\ -r\dot{r}\dot{\theta} - 0.5gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}T$$

Linearizing at r = 1.0m

$$\begin{bmatrix} 0.7 & 0 \\ 0 & 2.5 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -0.5g\theta \\ -0.5gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}T$$

In State-Space form

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -1.96 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix}T$$

The open-loop system is unstable

```
>> A = [0,0,1,0 ; 0,0,0,1 ; 0,-7,0,0 ; -1.96,0,0,0]

A =

         0         0    1.0000         0
         0         0         0    1.0000
         0   -7.0000         0         0
   -1.9600         0         0         0

>> eig(A)

ans =

  -1.9246
   0.0000 + 1.9246i
   0.0000 - 1.9246i
   1.9246

>>
```