ECE 463/663 - Homework #9

Calculus of Variations. LQG Control. Due Monday, April 7th

Soap Film

1) Calculate the shape of a soap film connecting two rings around the X axis:
   - Y(0) = 6
   - Y(2) = 5

The shape of a soap film minimizes the following functional

$$F = y\sqrt{1 + (y')^2}$$

From the lecture notes, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

Plugging in the two endpoints gives 2 equations for 2 unknowns

$$6 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

$$5 = a \cdot \cosh\left(\frac{2-b}{a}\right)$$

Solving using fminsearch and Matlab - first create a cost function

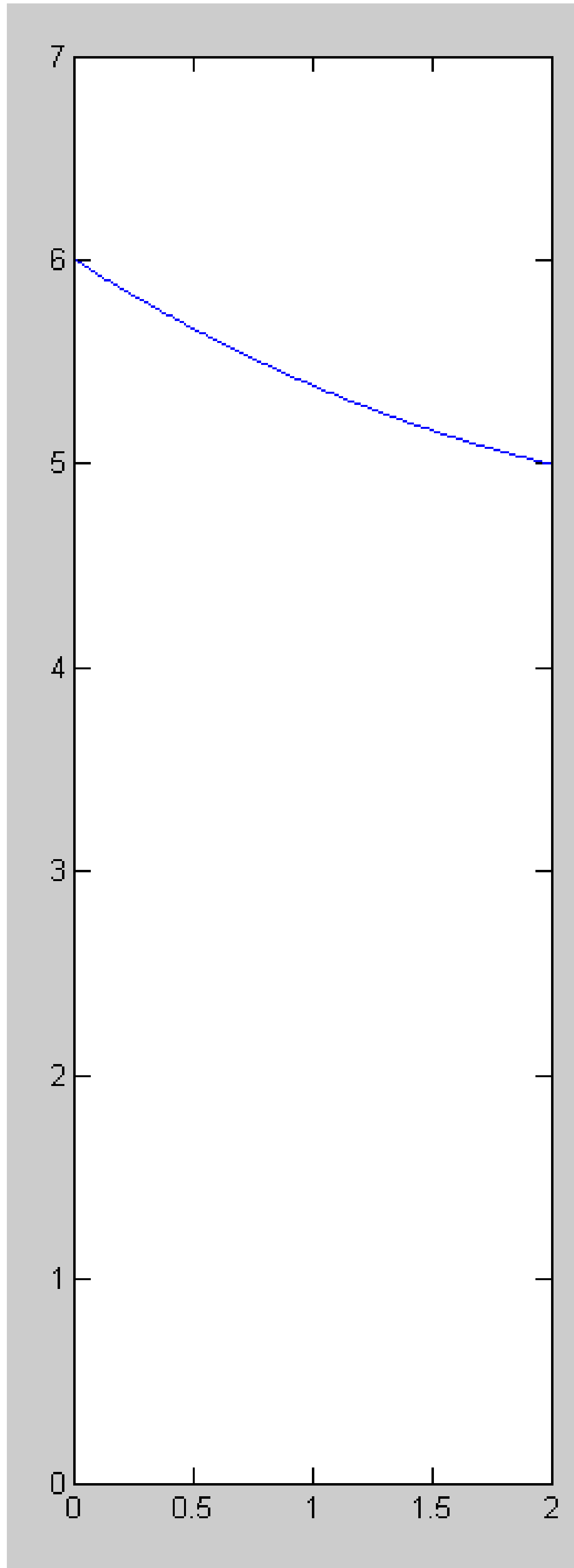
```
function [ J ] = cost_soap( z )

   a = z(1);
   b = z(2);

   e1 = a * cosh((0-b)/a) - 6;
   e2 = a * cosh((2-b)/a) - 5;

   J = e1^2 + e2^2;

   end
```

Now solve using Matlab:

```
>> [Z,e] = fminsearch('cost_soap',[1,2])

          a          b
Z =    4.8223    3.3052

e =  5.2301e-010

>>
```

Plotting:

```
>> a = Z(1);
>> b = Z(2);
>> x = [0:0.01:2]';
>> y = a*cosh((x-b)/a);
>> plot(x,y);
>> ylim([0,7])
>>
```



2) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 6
- Y(2) = free

The shape of a soap film minimizes the following functional

$$F = y\sqrt{1+(y')^2}$$

From the lecture notes, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

Plugging in the left endpoint

$$6 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

The right endpoint satisfies the constraint

$$F_{y'} = 0$$

$$-a\sinh\left(\frac{x-b}{a}\right) = 0$$

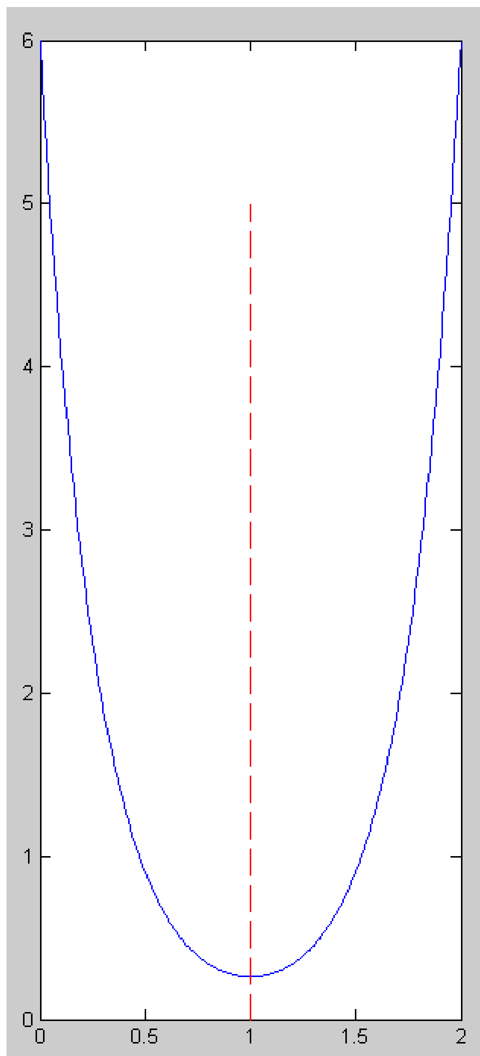
$$\sinh\left(\frac{2-b}{a}\right) = 0$$

The cost function in Matlab becomes

```
function [ J ] = cost_soap( z )

   a = z(1);
   b = z(2);

   e1 = a * cosh((0-b)/a) - 6;
   e2 = -a * sinh((1-b)/a);

   J = e1^2 + e2^2;

   end
```

Minimizing it

```
>> [Z,e] = fminsearch('cost_soap',[1,2])

              a          b
Z =      0.2614     1.0000

e =   1.9957e-009

>> a = Z(1);
>> b = Z(2);
>> x = [0:0.01:2]';
>> y = a*cosh((x-b)/a);
>> plot(x,y,'b',[1,1],[0,5],'r--')
>>
```



Hanging Chain

3) Calculate the shape of a hanging chain subject to the following constraints

- Length of chain = 4 meters
- Left Endpoint: (0,6)
- Right Endpoint: (2,5)

The functional a hanging chain minimizes is

$$F = x\sqrt{1+(y')^2} + M\sqrt{1+(y')^2}$$

The solution (from lecture notes) is

$$y = a\cosh\left(\tfrac{x-b}{a}\right) - M$$

Plugging in the endpoint constraints

$$6 = a\cosh\left(\tfrac{0-b}{a}\right) - M$$

$$5 = a\cosh\left(\tfrac{2-b}{a}\right) - M$$

The length constraint gives

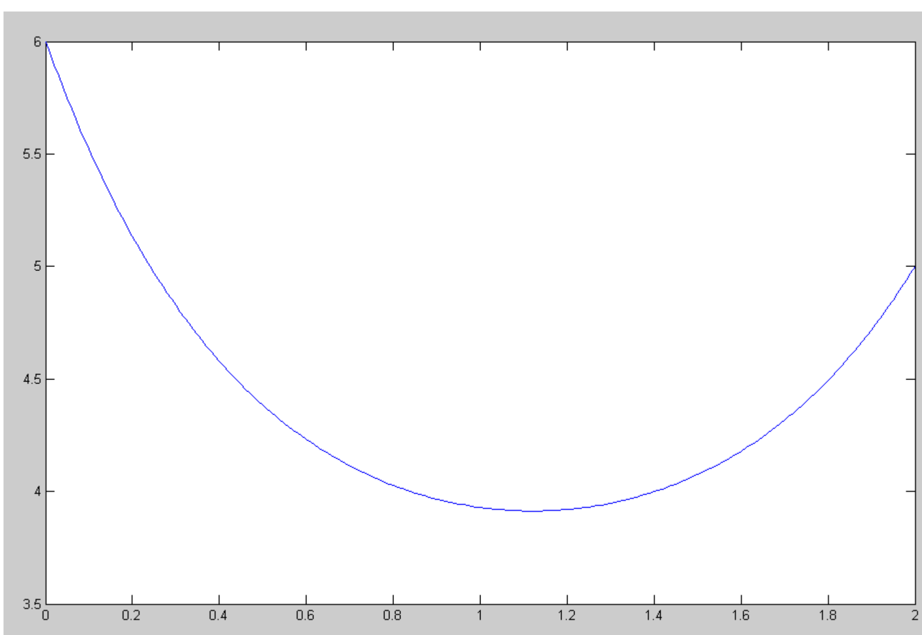
$$\left(a\sinh\left(\tfrac{x-b}{a}\right)\right)_0^2 = 4$$

Solving using Matlab

```
function [ J ] = cost_roap( z )

   a = z(1);
   b = z(2);
   M = z(3);

   e1 = a * cosh((0-b)/a) - M - 6;
   e2 = a * cosh((2-b)/a) - M - 5;
   e3 = a*sinh((2-b)/a) - a*sinh((0-b)/a) - 4;

   J = e1^2 + e2^2 + e3^2;

   x = [0:0.01:2];
   y = a*cosh( (x-b)/a ) - M;
   plot(x,y);
    pause(0.01);

   end


>> [Z,e] = fminsearch('cost_roap',[1,2,3])

           a          b         M
Z =    0.4717    1.1205   -3.4415

e =  1.2132e-008
```

Result:

$$y = 0.4717 \cdot \cosh\left(\frac{x-1.1205}{0.4717}\right) + 3.4416$$



Ricatti Equation

4) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^{10} (x^2 + 9\dot{x}^2)dt \qquad x(0) = 6 \qquad x(10) = 4$$

The funcitonal is

$$F = x^2 + 9\dot{x}^2$$

Solving the Euler LaGrange equation

$$\frac{d}{dt}(F_{x'}) - F_x = 0$$

$$\frac{d}{dt}(18\dot{x}) - 2x = 0$$

$$18\ddot{x} - 2x = 0$$

Using LaPlace notation

$$9s^2X - X = 0$$

$$s = \pm\frac{1}{3}$$

$$x(t) = ae^{t/3} + be^{-t/3}$$

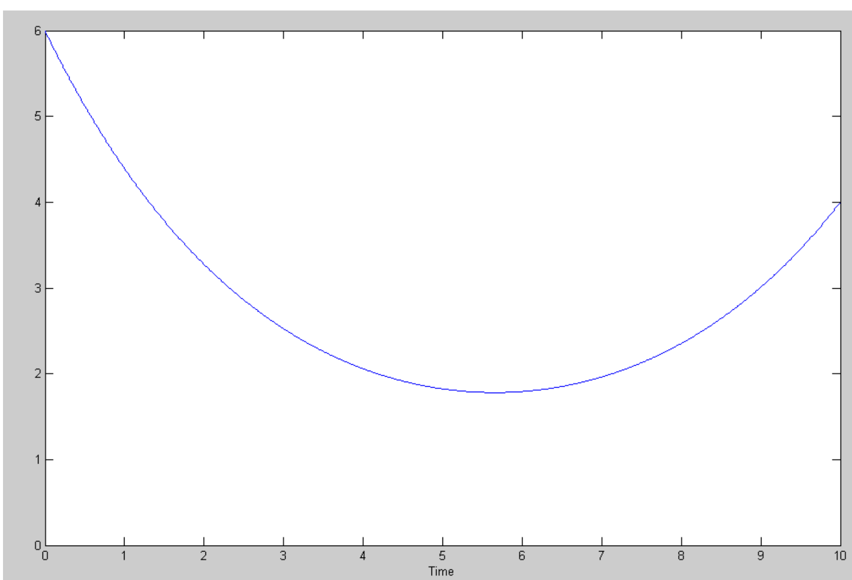
Plugging in the endpoint constraints

$$x(0) = 6 = a + b$$

$$x(10) = 4 = 28.03a + 0.03567b$$

Solving

a = 0.1352 b = 5.8648



5) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^8 (4x^2 + 9u^2)dt$$

$$\dot{x} = -0.2x + u$$

x(0) = 6

x(10) = 4

The functional is

$$F = 4x^2 + 9u^2 + m(-0.2x + u - \dot{x})$$

This results in three Euler LaGrange equations

With respect to x

$$\tfrac{d}{dt}(F_{x'}) - F_x = 0$$

$$\tfrac{d}{dt}(-m) - (8x - 0.2m) = 0$$

$$-\dot{m} - 8x + 0.2m = 0$$

With respect to u

$$\tfrac{d}{dt}(F_{u'}) - F_u = 0$$

$$\tfrac{d}{dt}(0) - (18u + m) = 0$$

With respect to m

$$\tfrac{d}{dt}(F_{m'}) - F_m = 0$$

$$-(-0.2x + u - \dot{x}) = 0$$

Solving

$$m = -18u$$

$$u = \dot{x} + 0.2x$$

$$-\dot{m} - 8x + 0.2m = 0$$

$$-(-18\dot{u}) - 8x + 0.2(-18u) = 0$$

$$18(\ddot{x} + 0.2\dot{x}) - 8x - 3.6(\dot{x} + 0.2x) = 0$$

$$\ddot{x} - 0.4844x = 0$$

Using LaPlace notation

$$(s^2 - 0.4844)X - 0$$

$$s = \pm 0.6960$$

and

$$x(t) = ae^{0.6960t} + be^{-0.6960t}$$

Plugging in the endpoints

$$x(0) = 6 = a + b$$

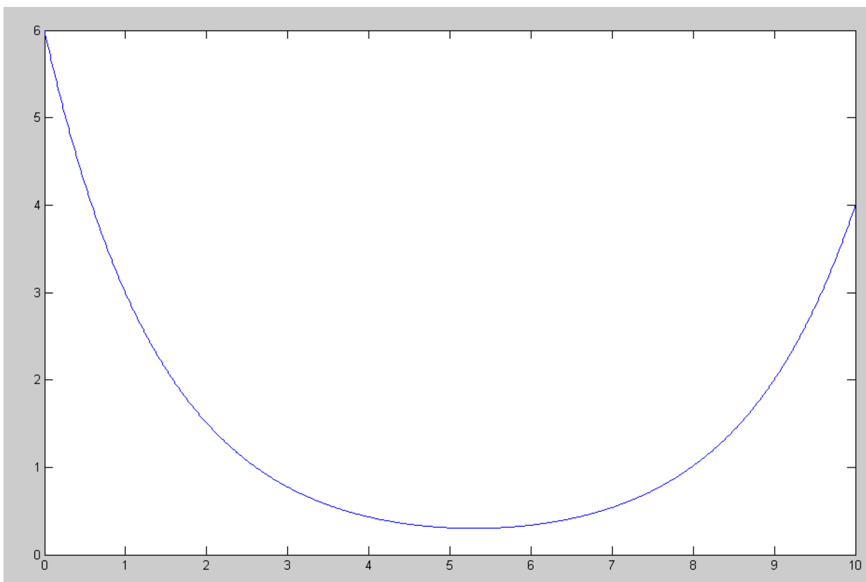
$$x(10) = 4 = 1053.8a + 0.000949b$$

solving

a = 0.0038

b = 5.9962

$$x(t) = 0.0038 \cdot e^{0.6960t} + 5.9962 \cdot e^{-0.6969t}$$



LQG Control

6) Cart & Pendulum (HW #6): Design a full-state feedback control law of the form

$$U = K_r R - K_x X$$

for the cart and pendulum system from homework #6 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

The desired closed-loop dominant pole should be at about

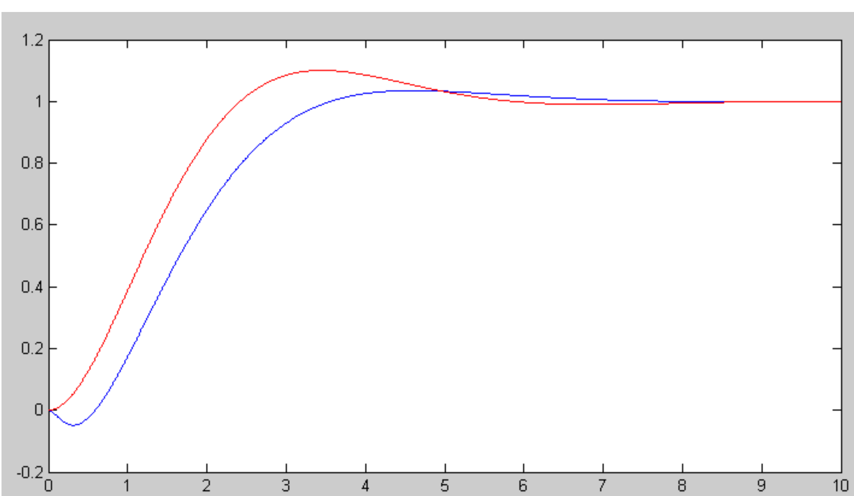
$$s = -0.67 + j0.91$$

for the desired transfer function (2nd order approximation)

$$G_d = \left(\frac{1.27}{s^2+1.33s+1.27} \right)$$

It's kind of subjective, but what I would up with is

```
Kx = lqr(A, B, diag([30,0,0,0]), 1);
DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;

Kx =   -5.4772 -126.2762   -9.2941  -24.8375

>> eig(A - B*Kx)

  -7.0000 + 0.3135i
  -7.0000 - 0.3135i
  -0.7717 + 0.7055i
  -0.7717 - 0.7055i

Kr =   -5.4772
```



Compare your results with homework #6

- Where are the closed-loop poles with pole placement and with LQG control?
- Are the feedback gains larger or smaller with LQG control?
- Which one works better?

In homework #6, I placed the poles

```
>> Kx = ppl(A, B, [-0.5+j*0.54, -0.5-j*0.54, -3, -4])

Kx =   -0.6632  -69.2048   -1.6113   -9.6113
```

The gains are similar - slightly larger with LQR. But then the system is slightly faster (dominant poles at -0.77 + j0.7)

Changing Q so that the dominant pole is about the same spot

```
>> Kx = lqr(A, B, diag([6,0,0,0]), 1);


Kx =   -2.4495 -115.0882   -5.7478  -20.7570

eig(A - B*Kx)


  -7.0000 + 0.1400i
  -7.0000 - 0.1400i
  -0.5046 + 0.4848i
  -0.5046 - 0.4848i
```

For a "fair" comparison, LQR gave slightly larger gains. They're pretty close however

7) Ball and Beam (HW #6): Design a full-state feedback control law of the form

$$U = K_r R - K_x X$$

for the ball and beam system from homework #6 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

```
Kx = lqr(A, B, diag([1,400,0,0]), 1);
Kx =  -9.9010   38.7983   -7.4456   13.9281

DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
Kr =   -5.0010

>> eig(A - B*Kx)

ans =

  -1.6553 + 1.6553i
  -1.6553 - 1.6553i
  -1.1303 + 1.1303i
  -1.1303 - 1.1303i
```

in contrast with pole placement

```
>> Kx = ppl(A, B, [-0.5+j*0.54, -0.5-j*0.54, -3, -4])

Kx =   -7.2211   48.8540   -5.6397   20.0000
```