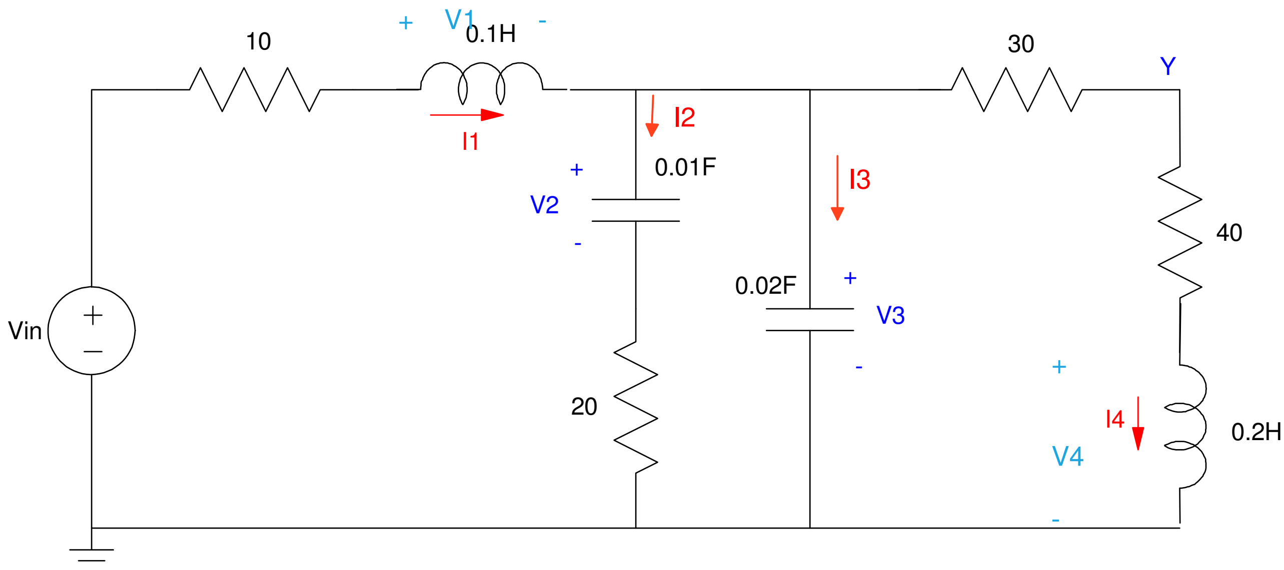
ECE 463/663 - Homework #2

State-Space, Eigenvalues, Eigenvectors. Due Monday, Jan 26th

Please make the subject "ECE 463/663 HW#2" if submitting homework electronically to Jacob_Glower@yahoo.com (or on blackboard)

1) For the following RLC circuit

- Specify the dynamics for the system (write N coupled differential equations)
- Express these dynamics in state-space form
- Determine the transfer function from Vin to Y



Step 1: Write the differential equations

$$V_1 = 0.1sI_1 = V_{in} - 10I_1 - V_3$$

$$I_2 = 0.01sV_2 = \left(\frac{V_3 - V_2}{20}\right)$$

$$I_3 = 0.02sV_3 = I_1 - I_4 - \left(\frac{V_3 - V_2}{20}\right)$$

$$V_4 = 0.2sI_4 = V_3 - 70I_4$$

$$Y = V_3 - 30I_4$$

Group terms and simplify

$$sI_1 = 10V_{in} - 100I_1 - 10V_3$$

$$sV_2 = 2.5V_3 - 2.5V_2$$

$$sV_3 = 50I_1 - 50I_4 - 2.5V_3 + 2.5V_2$$

$$sI_4 = 5V_3 - 350I_4$$

$$Y = V_3 - 30I_4$$

Place in state-space (matrix) form

$$\begin{bmatrix} sI_1 \\ sV_2 \\ sV_3 \\ sI_4 \end{bmatrix} = \begin{bmatrix} -100 & 0 & -10 & 0 \\ 0 & -2.5 & 2.5 & 0 \\ 50 & 2.5 & -2.5 & -50 \\ 0 & 0 & 5 & -350 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ V_3 \\ I_4 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & -30 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ V_3 \\ I_4 \end{bmatrix}$$

Using Matlab to find the transfer function

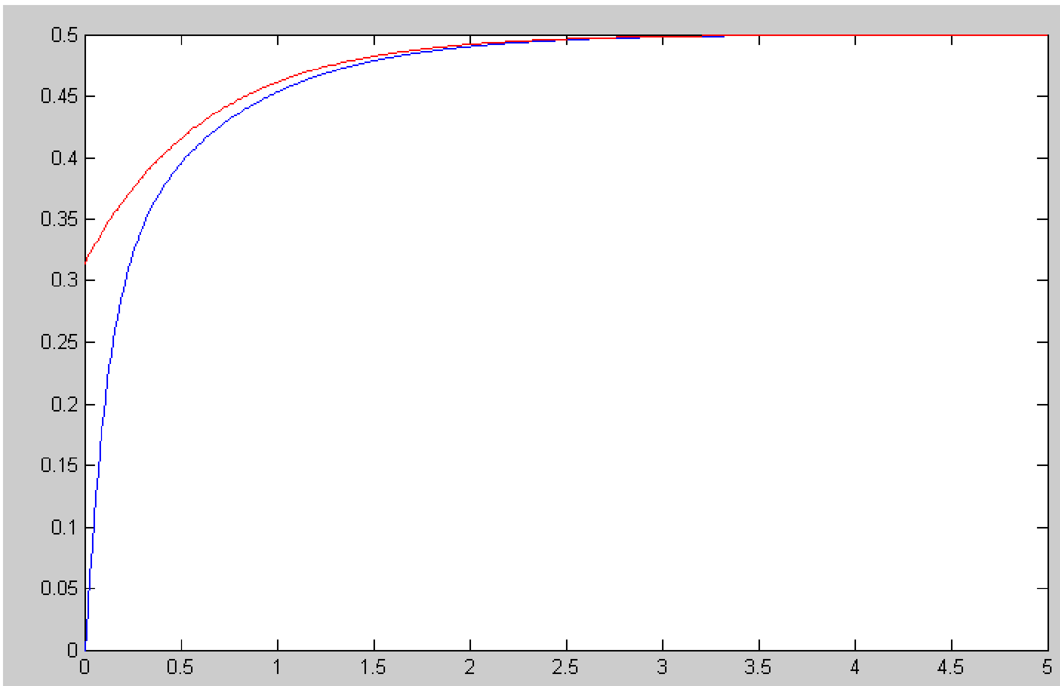
```
>> A = [-100,0,-10,0 ; 0,-2.5,2.5,0 ; 50,2.5,-2.5,-50 ; 0,0,5,-350]

 -100.0000         0  -10.0000         0
         0   -2.5000    2.5000         0
   50.0000    2.5000   -2.5000  -50.0000
         0         0    5.0000 -350.0000

>> B = [10;0;0;0]

    10
     0
     0
     0

>> C = [0,0,1,-30]

     0     0     1   -30

>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

         500 (s+2.5) (s+200)
--------------------------------------
(s+349.3) (s+94.5) (s+9.643) (s+1.571)
```

2) For the transfer function from Vin to Y

- Determine a 1st or 2nd-order approximation for this trasfer function
- Plot the step response of the actual 4th-order system and its approximation

```
>> DC = evalfr(G,0)

DC =    0.5000

>> G1 = zpk([],-1.571,1.571*0.5);
>> t = [0:0.01:5]';
>> y4 = step(G,t);
>> y1 = step(G1,t);
>> plot(t,y4,t,y1);
>> G1 = zpk(-2.5,-1.571,0.5*1.571/2.5)

0.3142 (s+2.5)
--------------
  (s+1.571)

>> y4 = step(G,t);
>> y1 = step(G1,t);
>> plot(t,y4,'b',t,y1,'r');
```

>>



4th-order system (red) and its 1st-order approximation (blue)

3) For this circuit...

- What initial condition will the energy in the system decay as slowly as possible?
- What initial condition will the energy in the system decay as fast as possible?

This is an eigenvector problem

```
>> [M,V] = eig(A)

M =

I1   -0.0057   -0.8762   -0.1039    0.0354
V2    0.0010   -0.0131   -0.3285   -0.9368
V3   -0.1419    0.4817    0.9387   -0.3482
I4   -0.9899    0.0094    0.0138   -0.0050


V =

 -349.2832         0         0         0
         0  -94.5029         0         0
         0         0   -9.6431         0
         0         0         0   -1.5708
```

>>

Fast Mode: Make the initial conditon the 1st eigenvector (red). That results in the initial conditions decaying as exp(-349t)

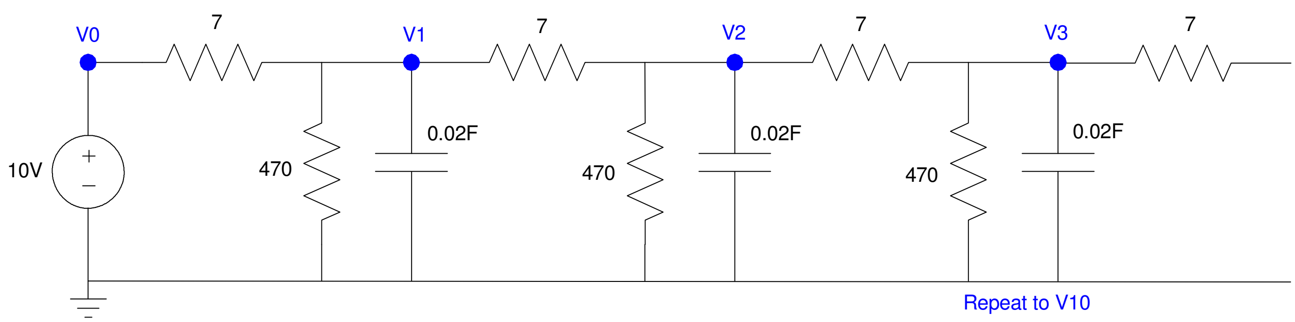
Slow Mode: Make the initial conditon the 4th eigenvector (blue). That results in the initial conditions decaying as exp(-1.57t)

Problem 4-7: 10-Stage RC Filter.

note: You can turn in the Matlab code along with screen shots of the plots if you like.

4) For the following 10-stage RC circuit

- Specify the dynamics for the system (write N coupled differential equations)
  - note: Nodes 1..9 have the same form. Just write the node equation for node 1 and node 10.
- Express these dynamics in state-space form
- Determine the transfer function from Vin to V10



At node V1:

$$0.02sV_1 = \left(\frac{V_0 - V_1}{7}\right) + \left(\frac{V_2 - V_1}{7}\right) - \left(\frac{V_1}{470}\right)$$

Simplifying (the same pattern holds for nodes 1..9)

$$sV_1 = 7.143V_0 - 14.288V_1 + 7.143V_2$$

$$sV_2 = 7.143V_1 - 14.288V_2 + 7.143V_3$$

$$\vdots$$

$$sV_9 = 7.143V_8 - 14.288V_9 + 7.143V_{10}$$

Node #10 is slightly different since it's missing a 7-ohm resistor to the right

$$sV_{10} = 7.143V_9 - 7.145V_{10}$$

Put this into matrix form and then into Matlab

```
>> A = zeros(10,10);
>> for i=1:9
A(i,i) = -14.288;
A(i+1,i) = 7.143;
A(i,i+1) = 7.143;
end
>> A(10,10) = -7.145;
>> A

A =

  -14.2880    7.1430         0         0         0         0         0         0         0         0
    7.1430  -14.2880    7.1430         0         0         0         0         0         0         0
         0    7.1430  -14.2880    7.1430         0         0         0         0         0         0
         0         0    7.1430  -14.2880    7.1430         0         0         0         0         0
         0         0         0    7.1430  -14.2880    7.1430         0         0         0         0
         0         0         0         0    7.1430  -14.2880    7.1430         0         0         0
         0         0         0         0         0    7.1430  -14.2880    7.1430         0         0
         0         0         0         0         0         0    7.1430  -14.2880    7.1430         0
         0         0         0         0         0         0         0    7.1430  -14.2880    7.1430
         0         0         0         0         0         0         0         0    7.1430   -7.1450

>> B = zeros(10,1);
>> B(1) = 7.143;
>> B

B =

    7.1430
         0
         0
         0
         0
         0
         0
         0
         0
         0

>> C = zeros(1,10);
>> C(10) = 1;
>> C

C =

     0     0     0     0     0     0     0     0     0     1

>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

Zero/pole/gain:
                                  345785279.7854
-----------------------------------------------------------------------------------------------
(s+27.94) (s+26.09) (s+23.2) (s+19.51) (s+15.36) (s+11.11) (s+7.145) (s+3.816) (s+1.417) (s+0.1616)

>>
```

5) For the transfer function for problem #4

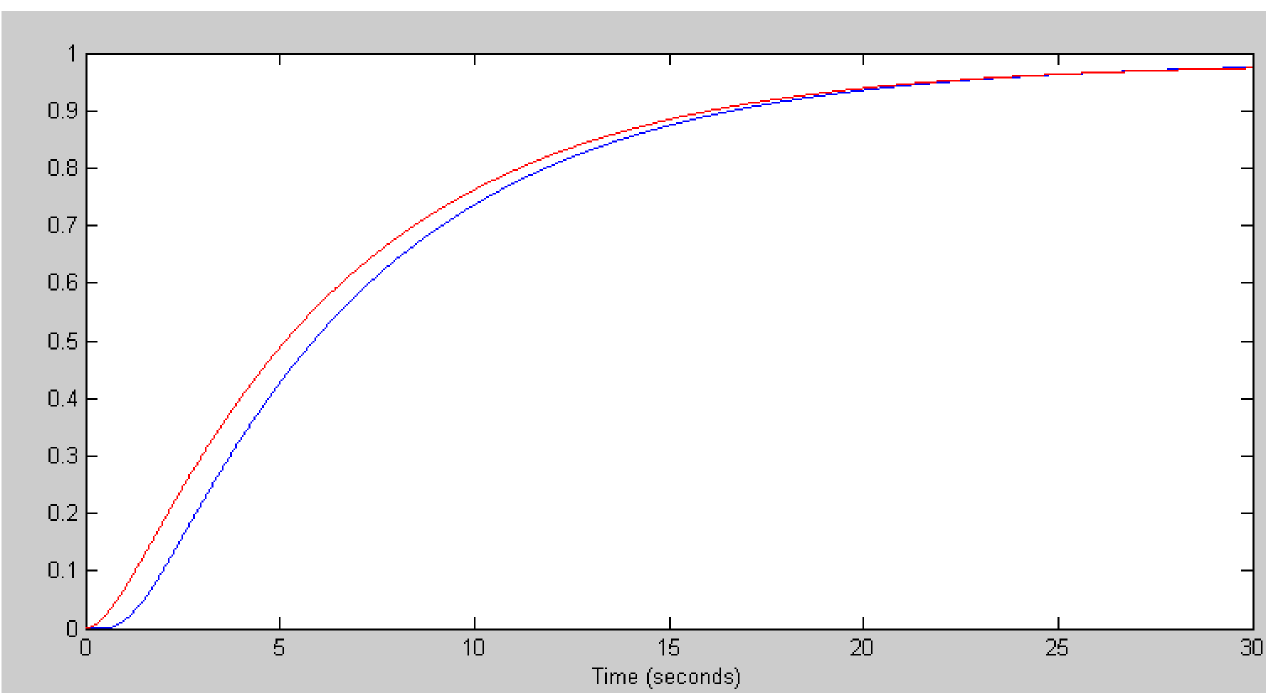
- Determine a 2nd-order approximation for this trasfer function
- Plot the step response of the actual 10th-order system and its 2nd-order approximation

The 10th-order system is

```
                                        345785279.7854
---------------------------------------------------------------------------------------------------
(s+27.94) (s+26.09) (s+23.2) (s+19.51) (s+15.36) (s+11.11) (s+7.145) (s+3.816) (s+1.417) (s+0.1616)
```

Keep the two dominant poles and match the DC gain

```
>> G2 = zpk([],[-0.1616,-1.417],1);
>> G10 = G;
>> k = evalfr(G10,0) / evalfr(G2,0)

k =    0.2255

>> G2 = zpk([],[-0.1616,-1.417],0.225)


       0.225
--------------------
(s+0.1616) (s+1.417)

>>
>> t = [0:0.01:30]';
>> y10 = step(G10,t);
>> y2 = step(G2,t);
>> plot(t,y10,'b',t,y2,'r');
>> xlabel('Time (seconds)')
```



Step response of the 10th-order system (blue) and its 2nd-order approximation (red)

6) For the circuit for problem #4

- What initial condition will decay as slowly as possible?
- What initial condition will decay as fast as possible?

This is an eigenvector problem

```
>> [M,V] = eig(A)

M =

   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
    0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412    0.0000    0.3412   -0.4255    0.1894
    0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
   -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255   -0.0000   -0.4255   -0.1894    0.3412
   -0.3780    0.3780    0.0000    0.3780   -0.3780    0.0000    0.3780   -0.3780    0.0000    0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
    0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352


  -27.9393  -26.0916  -23.1952  -19.5073  -15.3556  -11.1091   -7.1450   -3.8156   -1.4168   -0.1616

>>
```

Fast Mode: If the initial condition is the 1st eigenvector (red), the transients will decay as exp(-27.93t)

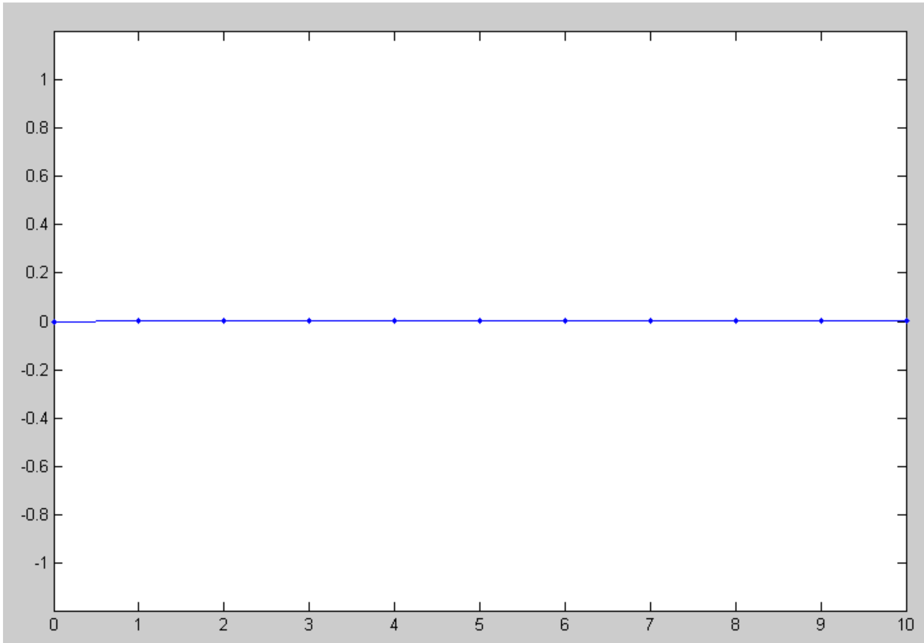
Slow Mode: If the initial condition is the 10th eigenvector (blue), the transients will devay as exp(-0.16t)

7) Modify the program *heat.m* to match the dynamics you calculated for this problem.

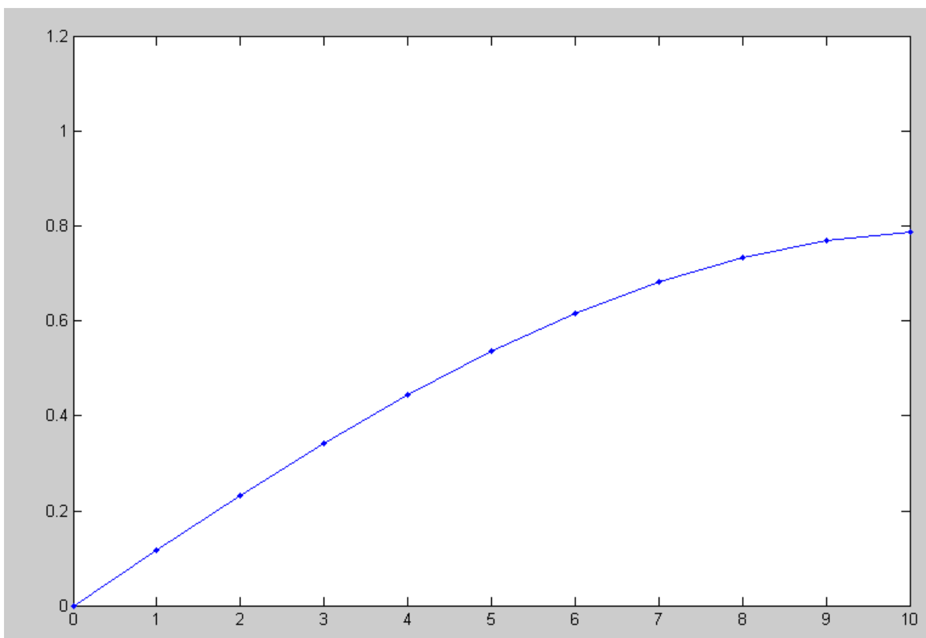
- Give the program listing
- Give the response for Vin = 0 and the initial conditions being
  - The slowest eigenvector
  - The fastest eigenvector
  - A random set of voltages

Code:

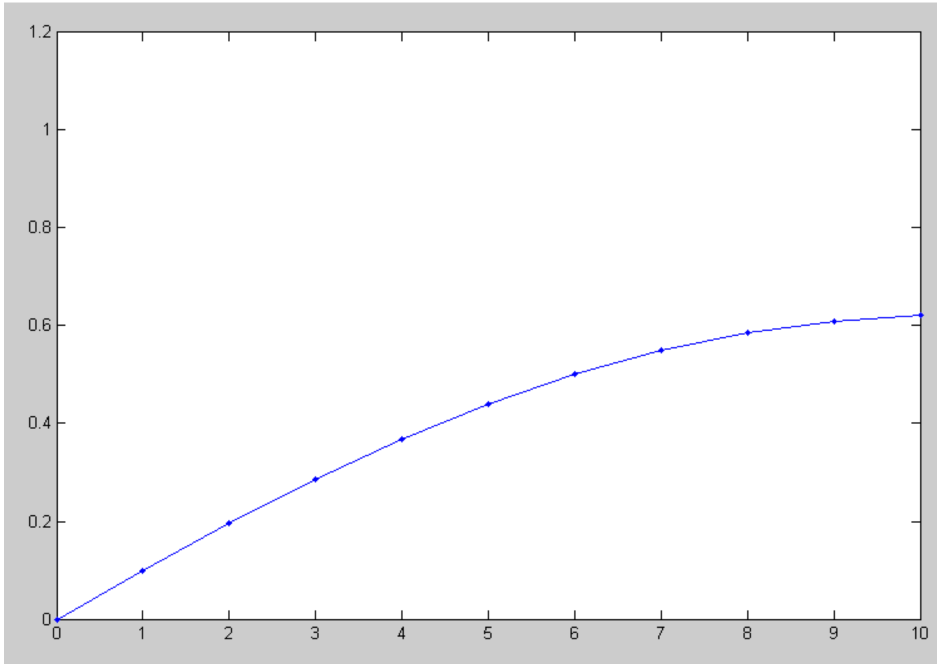
```
% 10-stage RC Filter

V = rand(10,1);
dV = 0*V;

dt = 0.01;
t = 0;
V0 = 0;

REF = 1;
Z = 0;

DATA = [];

while(t < 2)

   dV(1) = 7.143*V0   - 14.288*V(1) + 7.143*V(2);
   dV(2) = 7.143*V(1) - 14.288*V(2) + 7.143*V(3);
   dV(3) = 7.143*V(2) - 14.288*V(3) + 7.143*V(4);
   dV(4) = 7.143*V(3) - 14.288*V(4) + 7.143*V(5);
   dV(5) = 7.143*V(4) - 14.288*V(5) + 7.143*V(6);
   dV(6) = 7.143*V(5) - 14.288*V(6) + 7.143*V(7);
   dV(7) = 7.143*V(6) - 14.288*V(7) + 7.143*V(8);
   dV(8) = 7.143*V(7) - 14.288*V(8) + 7.143*V(9);
   dV(9) = 7.143*V(8) - 14.288*V(9) + 7.143*V(10);
   dV(10) = 7.143*V(9) - 7.145*V(10);

   V = V + dV * dt;
   t = t + dt;

   plot([0:10], [V0;V], 'b.-');
   ylim([0,1.2]);
   xlim([0,10]);
   pause(0.01);
   DATA = [DATA ; V'];
end

%pause(2);
%plot(DATA)
```



Fast Eigenvector: Voltages at t = 2 seconds



Slow Eigenvector: Voltages at t = 2 seconds



Random Initial Condition: Voltages at t = 2 seconds