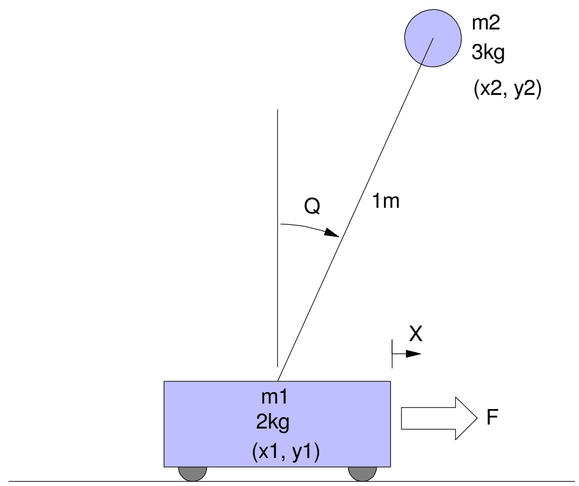
ECE 463/663 - Homework #4

Block Diagrams and LaGrangian Dynamics. Due Monday, February 10th

1) (30pt) Derive the dynamics for an inverted pendulum where

- m1 = 2kg (mass of cart)
- m2 = 3kg (mass of ball)
- L = 1.0m (length of arm)

Fine the linearized dynamics at x = 0, θ = 0



Mass #1

$$x_1 = x \qquad y_1 = 0$$

$$\dot{x}_1 = \dot{x} \qquad \dot{y}_1 = 0$$

$$PE = 0$$

$$KE = \tfrac{1}{2}mv^2 = \dot{x}^2$$

Mass #2

$$x_2 = x + \sin\theta \qquad y_2 = \cos\theta$$

$$\dot{x}_2 = \dot{x} + \cos\theta\dot{\theta} \qquad \dot{y}_2 = -\sin\theta\dot{\theta}$$

$$PE = m_2 g y_2 = 3g\cos\theta$$

$$KE = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$KE = \tfrac{3}{2}\left(\left(\dot{x} + \cos\theta\dot{\theta}\right)^2 + \left(-\sin\theta\dot{\theta}\right)^2\right)$$

$$KE = 1.5\dot{x}^2 + 1.5\dot{\theta}^2 + 3\cos\theta\dot{x}\dot{\theta}$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left(\dot{x}^2 + 1.5\dot{x}^2 + 1.5\dot{\theta}^2 + 3\cos\theta\dot{x}\dot{\theta}\right) - (3g\cos\theta)$$

To find the dynamics, use the Euler LaGrange equation

$$L = \left(2.5\dot{x}^2 + 1.5\dot{\theta}^2 + 3\cos\theta\dot{x}\dot{\theta}\right) - (3g\cos\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(5\dot{x} + 3\cos\theta\dot{\theta}\right) - (0)$$

$$F = 5\ddot{x} + 3\cos\theta\ddot{\theta} - 3\sin\theta\dot{\theta}^2$$

$$L = \left(2.5\dot{x}^2 + 1.5\dot{\theta}^2 + 3\cos\theta\dot{x}\dot{\theta}\right) - (3g\cos\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(3\dot{\theta} + 3\cos\theta\dot{x}\right) - \left(-3\sin\theta\dot{x}\dot{\theta} + 3g\sin\theta\right)$$

$$T = 3\ddot{\theta} + 3\cos\theta\ddot{x} - 3\sin\theta\dot{x}\dot{\theta} + 3\sin\theta\dot{x}\dot{\theta} - 3g\sin\theta$$

$$T = 3\ddot{\theta} + 3\cos\theta\ddot{x} - 3g\sin\theta$$

So, the dynamics are

$$F = 5\ddot{x} + 3\cos\theta\ddot{\theta} - 3\sin\theta\dot{\theta}^2$$

$$T = 3\ddot{\theta} + 3\cos\theta\ddot{x} - 3g\sin\theta$$

In Matrix form

$$\begin{bmatrix} 5 & 3\cos\theta \\ 3\cos\theta & 3 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ T \end{bmatrix} + \begin{bmatrix} 3\sin\theta\dot{\theta}^2 \\ 3g\sin\theta \end{bmatrix}$$

Linearizing about zero with T = 0

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}F + \begin{bmatrix} 0 \\ 3g\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} F + \begin{bmatrix} -1.5g\theta \\ 2.5g\theta \end{bmatrix}$$

Putting this in state-space form

$$s\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.5g & 0 & 0 \\ 0 & 2.5g & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

or

$$s\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -14.7 & 0 & 0 \\ 0 & 24.5 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ sx \\ s\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

The poles are at

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>> A = [0,0,1,0 ; 0,0,0,1 ; 0,-14.7,0,0 ; 0,24.5,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0  -14.7000         0         0
         0   24.5000         0         0

>> B = [0;0;0.5;-0.5]

         0
         0
    0.5000
   -0.5000

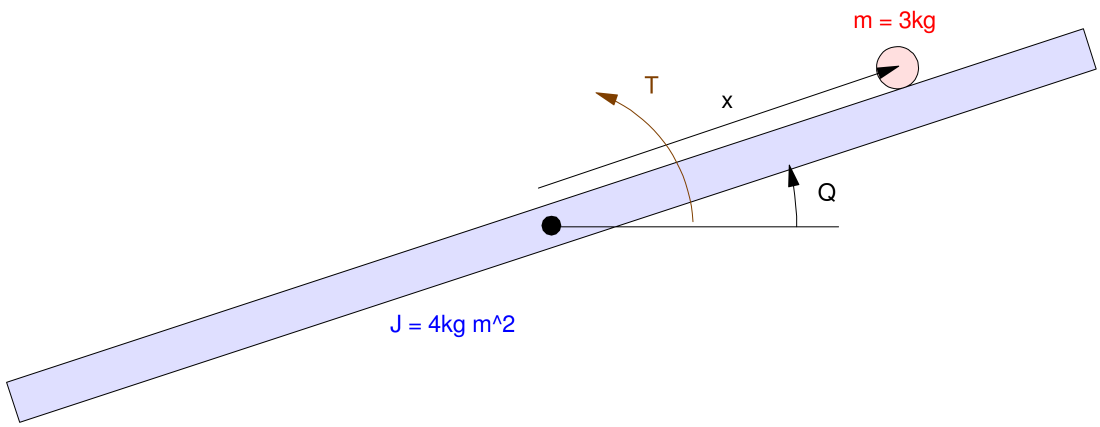
>> eig(A)

         0
         0
    4.9497
   -4.9497
```

2) (30pt) Derive the dynamics for a ball and beam system where

- J = 4.0 kg m2 (the inertia of the beam)
- m = 3kg (the mass of the ball)

Find the linearized dynamics at r = 1.0m, θ = 0



Position of the ball:

$$x_1 = r\cos\theta \qquad y_1 = r\sin\theta$$

$$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y}_1 = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

The potential and kinetic energy (assuming a solid sphere) are:

$$PE = mgy_1 = mgr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \tfrac{4}{2}\dot{\theta}^2 + \tfrac{1}{2}m\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = 2\dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = \left(2 + \tfrac{1}{2}mr^2\right)\dot{\theta}^2 + 0.7m\dot{r}^2$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left(\left(2 + \tfrac{1}{2}mr^2\right)\dot{\theta}^2 + 0.7m\dot{r}^2\right) - (mgr\sin\theta)$$

With m = 3kg

$$L = \left(2\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

Force on the Ball

$$L = \left(2\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial r'}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(2.1\dot{r}) - \left(3r\dot{\theta}^2 - 3g\sin\theta\right)$$

$$F = 1.2\ddot{r} - 3r\dot{\theta}^2 + 3g\sin\theta$$

Torque on the Beam

$$L = \left(2\dot{\theta}^2 + 1.5r^2\dot{\theta}^2 + 2.1\dot{r}^2\right) - (3gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(4\dot{\theta} + 3r^2\dot{\theta}\right) - (-3gr\cos\theta)$$

$$T = 4\ddot{\theta} + 3r^2\ddot{\theta} + 6r\dot{r}\dot{\theta} + 3gr\cos\theta$$

Putting it together

$$\begin{bmatrix} 1.2 & 0 \\ 0 & 4+3r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 3r\dot{\theta}^2 - 3g\sin\theta \\ -6r\dot{r}\dot{\theta} - 3gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearizing at r = 1.0m

$$\begin{bmatrix} 1.2 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -3g\theta \\ -3gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

In State-Space form

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -24.5 & 0 & 0 \\ -4.2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.143 \end{bmatrix} T$$

The open-loop system is unstable

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>> A = [0,0,1,0 ; 0,0,0,1 ; 0,-24.5,0,0 ; -4.2,0,0,0]

A =

         0         0    1.0000         0
         0         0         0    1.0000
         0  -24.5000         0         0
   -4.2000         0         0         0

>> eig(A)

ans =

  -3.1850
  -0.0000 + 3.1850i
  -0.0000 - 3.1850i
   3.1850
```