ECE 463/663 - Homework #3

Canonical Forms, Similarity Transforms, LaGrangian Dynamics, Block Diagrams
Due Monday, January 30th
Please submit as a hard copy or submit on BlackBoard

Canonical Forms

Problem 1-3) For the system

$$Y = \left(\frac{50(s+7.1)}{(s+2)(s+7)(s+10)}\right)U$$

1) Express this system in controller canonical form. (Give the A, B, C, D matrices)

Multiply it out

$$Y = \left(\frac{50s+355}{s^3+19s^2+104s+140}\right)U$$

By inspection

$$sX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -140 & -104 & -19 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 355 & 50 & 0 \end{bmatrix} X + [0]U$$

Checking in Matlab

```
>> A = [0,1,0;0,0,1;-140,-104,-19];
>> B = [0;0;1];
>> C = [355,50,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

     50 (s+7.1)
  ------------------
  (s+10) (s+7) (s+2)
```

2) Express this system in cascade form

Rewrite as

$$Y = \left(\left(\frac{a}{(s+2)(s+7)(s+10)} \right) + \left(\frac{b}{(s+2)(s+7)} \right) + \left(\frac{c}{(s+2)} \right) \right) U$$

$$Y = \left(\left(\frac{a}{(s+2)(s+7)(s+10)} \right) + \left(\frac{b(s+10)}{(s+2)(s+7)(s+10)} \right) + \left(\frac{c(s+7)(s+10)}{(s+2)(s+7)(s+10)} \right) \right) U$$

By inspection

c = 0

b = 50

a = -145

$$sX = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -7 & 0 \\ 0 & 1 & -10 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 50 & -145 \end{bmatrix} X$$

Checking in Matlab

```
>> A = [-2,0,0;1,-7,0;0,1,-10]

A =

    -2     0     0
     1    -7     0
     0     1   -10

>> B = [1;0;0];
>> C = [0,50,-145];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

    50 (s+7.1)
------------------
(s+10) (s+7) (s+2)
```

3) Express this system in Jordan (diagonal) form

Do a partial fraction expansion

$$Y = \left(\frac{50(s+7.1)}{(s+2)(s+7)(s+10)} \right) U$$

$$Y = \left(\left(\frac{6.375}{s+2} \right) + \left(\frac{-0.333}{s+7} \right) + \left(\frac{-6.0417}{s+10} \right) \right) U$$

By inspection

$$sX = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -10 \end{bmatrix} X + \begin{bmatrix} 6.375 \\ -0.333 \\ -6.0417 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$

Checking in Matlab

```
>> A = diag([-2,-7,-10])

A =

    -2     0     0
     0    -7     0
     0     0   -10

>> B = [6.375 ; -0.33333; -6.0417];
>> C = [1,1,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

-3e-005 (s-1.667e006) (s+7.1)
-----------------------------
     (s+10) (s+7) (s+2)

>> tf(G)


 -3e-005 s^2 + 50 s + 355
--------------------------
s^3 + 19 s^2 + 104 s + 140
```

A little rounding error but the same

4) Assume a system's dynamics are

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \\ sV_4 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 1 & -5 & 1 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} V_0$$

$$Y = V_3$$

Express these dynamic with the change in variable

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_3 - V_4 \\ V_1 + V_2 + V_3 + V_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = T^{-1}X$$

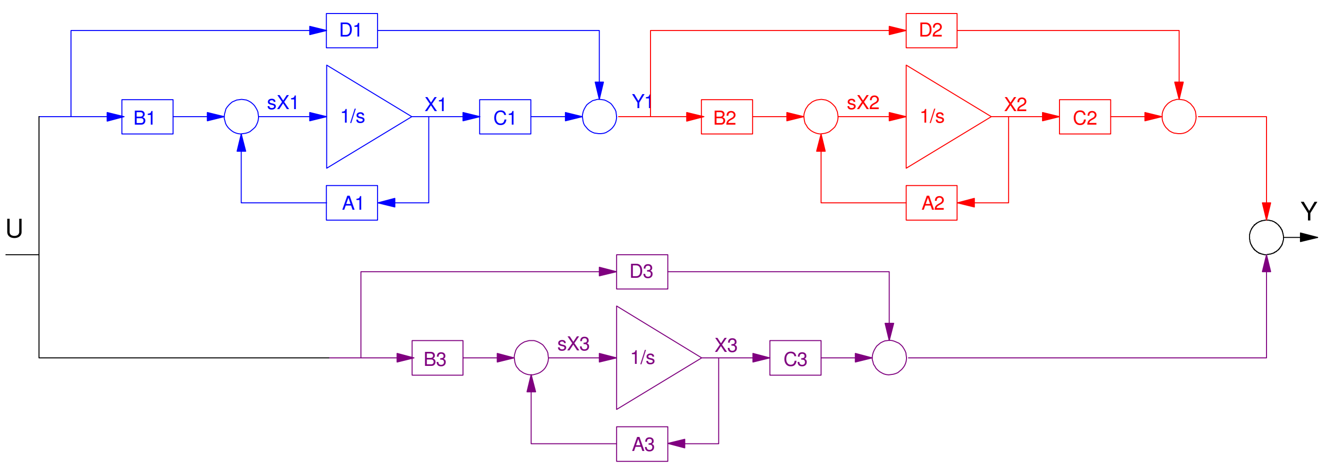
Using Matlab

```
>> A = [-5,1,0,0 ; 1,-5,1,0 ; 0,1,-5,1 ; 0,0,1,-5]

    -5     1     0     0
     1    -5     1     0
     0     1    -5     1
     0     0     1    -5

>> B = [1;2;3;4];
>> C = [0,0,1,0];
>> D = 0;
>> Ti = [1,-1,0,0;0,1,-1,0;0,0,1,-1;1,1,1,1]

Ti =

     1    -1     0     0
     0     1    -1     0
     0     0     1    -1
     1     1     1     1

>> T = inv(Ti)

    0.7500    0.5000    0.2500    0.2500
   -0.2500    0.5000    0.2500    0.2500
   -0.2500   -0.5000    0.2500    0.2500
   -0.2500   -0.5000   -0.7500    0.2500

>> Az = inv(T)*A*T

   -5.7500    0.5000   -0.2500   -0.2500
    1.0000   -5.0000    1.0000         0
   -0.2500    0.5000   -5.7500    0.2500
   -0.5000         0    0.5000   -3.5000

>> Bz = inv(T)*B

    -1
    -1
    -1
    10
```

```
>> Cz = C*T

Cz =

   -0.2500   -0.5000    0.2500    0.2500

>> Dz = D

Dz =

     0

>>
```

Block Diagrams

5) Determine the state-space model the following system:



Writing the equations

$$sX_1 = B_1 U + A_1 X_1$$

$$sX_2 = B_2 C_1 X_1 + B_2 D_1 U + A_2 X_2$$

$$sX_3 = B_3 U + A_3 X_3$$

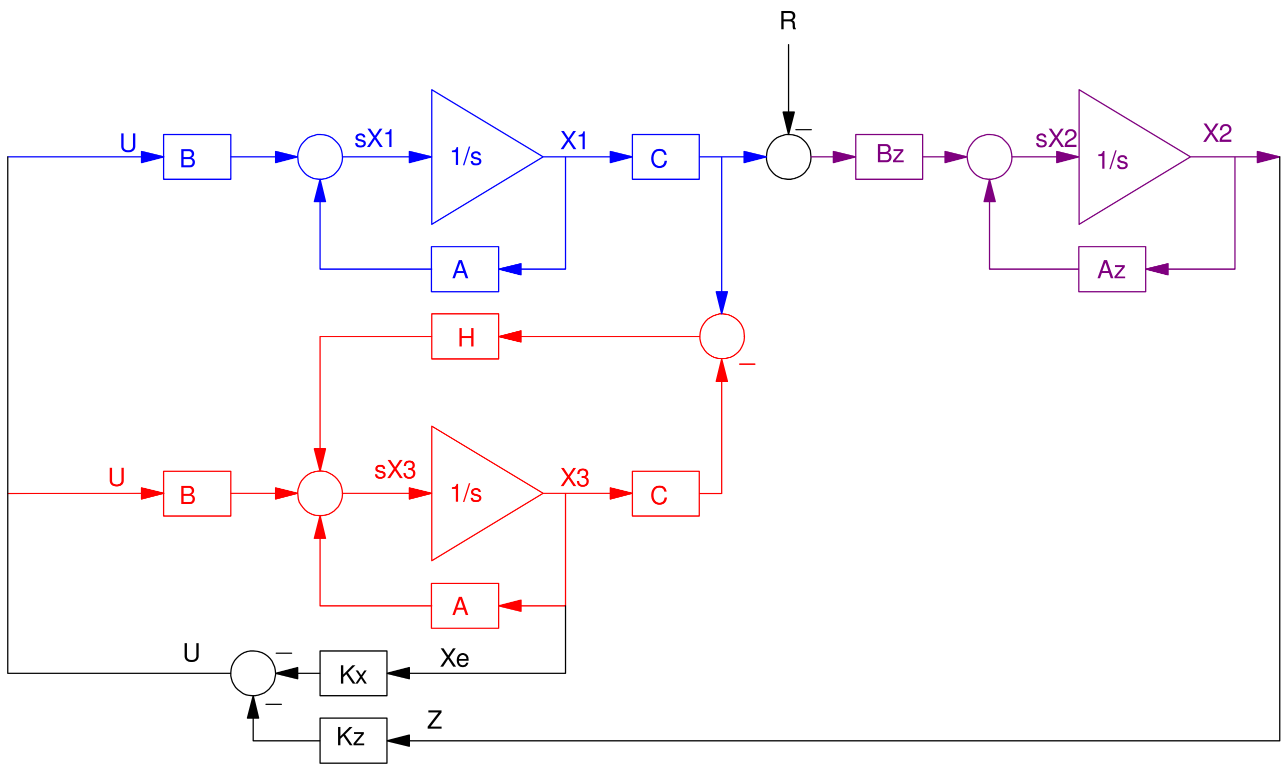
$$Y = C_2 X_2 + D_2 C_1 X_1 + D_2 D_1 U + C_3 X_3 + D_3 U$$

Placing in matrix form:

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ B_2 C_1 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \\ B_3 \end{bmatrix} U$$

$$Y = \begin{bmatrix} D_2 C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + [D_1 D_2 + D_3] U$$

6) Determine the state-space model for the following system:



Writing the equations

$$sX_1 = AX_1 - BK_xX_3 - BK_zX_2$$

$$sX_2 = A_zX_2 - B_zR + +B_zCX_1$$

$$sX_3 = AX_3 - BK_xX_3 - BK_zX_2 + HCX_1 - HCX_3$$

Grouping terms and placing in matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} (A) & (-BK_z) & (-BK_z) \\ (B_zC) & (A_z) & 0 \\ (HC) & (-BK_z) & (A - BK_x - HC) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -B_z \\ 0 \end{bmatrix} R$$

LaGrangian Dynamics

A 1kg ball is rolling on a surface defined by:

$$y = 0.2 \cdot (x-2)(x)(x+2)$$

$$y = 0.2(x^3 - 4x)$$

$$\dot{y} = (0.6x^2 - 0.8)\dot{x}$$

7) Determine the kinetic and potential energy of this ball as a function of x: Gravity is in the -y direction.

Assuming a solid sphere:

Potential Energy

$$PE = mgh = 0.2g(x^3 - 4x)$$

kinetic energy (solid sphere)

$$KE = 0.7m(\dot{x}^2 + \dot{y}^2)$$

$$KE = 0.7\left(\dot{x}^2 + ((0.6x^2 - 0.8)\dot{x})^2\right)$$

$$KE = 0.7(1 + 0.36x^4 - 0.96x^2 + 0.64)\dot{x}^2$$

8) Determine the dynamics for this ball as it rolls on this surface

The LaGrangian is then

$$L = KE - PE$$

$$L = 0.7(1 + 0.36x^4 - 0.96x^2 + 0.64)\dot{x}^2 - 0.2g(x^3 - 4x)$$

$$L = (0.252x^4 - 0.672x^2 + 1.148)\dot{x}^2 - 0.2g(x^3 - 4x)$$

The force on the ball is

$$F = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

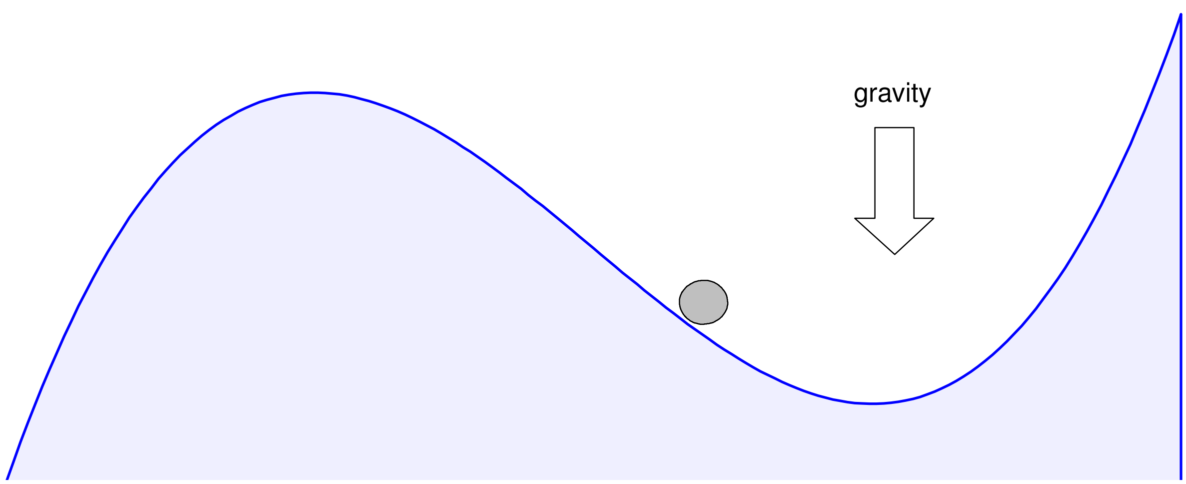
$$0 = \frac{d}{dt}((0.504x^4 - 1.344x^2 + 2.296)\dot{x})$$
$$-(1.008x^3 - 1.344x)\dot{x}^2 - 0.6gx^2 + 0.8g$$

$$0 = (0.504x^4 - 1.344x^2 + 2.296)\ddot{x}$$
$$+(2.0016x^3 - 2.688x)\dot{x}^2$$
$$-1.008x^3\dot{x}^2 + 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g$$

$$0 = \left(0.504x^4 - 1.344x^2 + 2.296\right)\ddot{x} + 1.008x^3\dot{x}^2 - 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g$$

The acceleration on the ball is then

$$\ddot{x} = -\left(\frac{1.008x^3\dot{x}^2 - 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g}{0.504x^4 - 1.344x^2 + 2.296}\right)$$



Code:

```
x = 0;
dx = 0;
g = -9.8;

dt = 0.01;
t = 0;
L = 0;
while(abs(x) < 3)

% Compute the dynamics
 num = (1.008*x^3 - 1.344*x)*(dx^2) - 0.6*g*x^2 + 0.8*g;
 den = 0.504*x^4 - 1.344*x^2 + 2.296;

 ddx = - num / den;

% Integrate
 x  = x  + dx  * dt;
 dx = dx + ddx * dt;
 dy = (0.6*x^2 - 0.8)*dx;
 dL = sqrt(1 + (dy/dx)^2) * dx;
 L = L + dL * dt;
 t  = t  + dt;

% Display the ball
 y = 0.2 * x * (x-2) * (x+2);

 x1 = [-3:0.01:3]';
 y1 = 0.2 * x1 .* (x1-2) .* (x1+2);


 % draw the ball
 i = [0:0.01:1]' * 2 * pi;
 xb = 0.05*cos(i) + x;
 yb = 0.05*sin(i) + y + 0.05;

 % line through the ball
 q = [0, pi] - L/0.05;
 xb1 = 0.05*cos(q) + x;
 yb1 = 0.05*sin(q) + y + 0.05;

 plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');
 xlim([-3,3]);
 ylim([-2,2]);
 pause(0.01);
 end
```