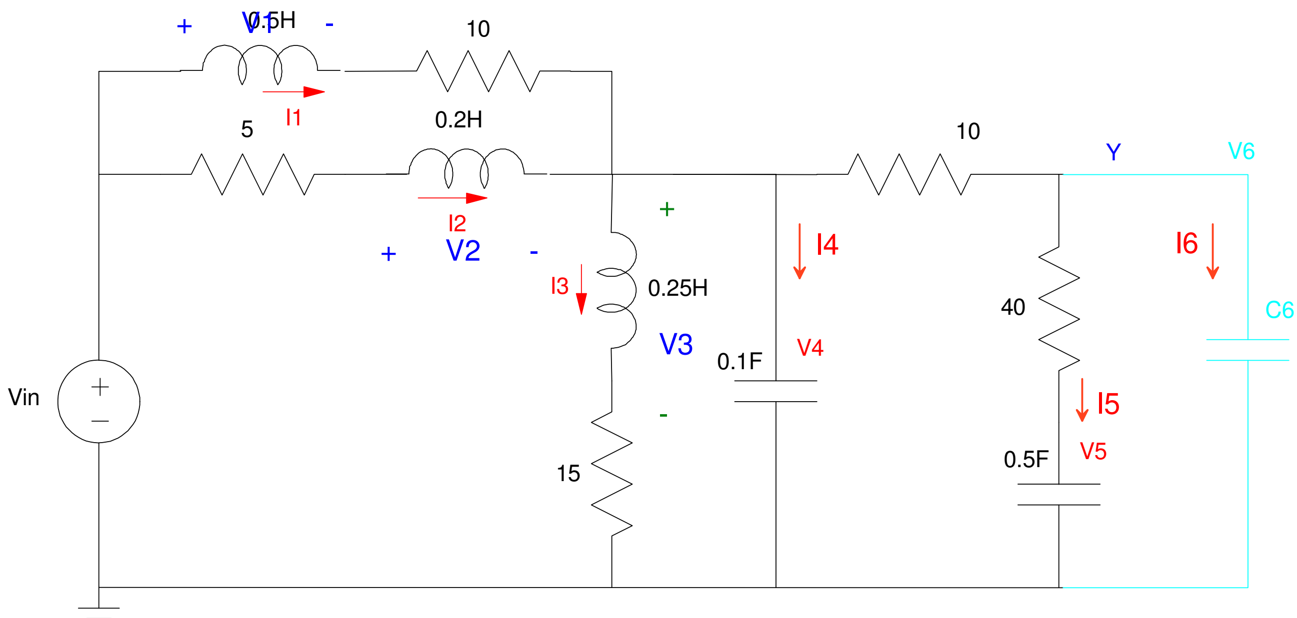
ECE 463/663 - Homework #2

State-Space, Eigenvalues, Eigenvectors. Due Monday, Jan 22nd
Please submit as a hard copy or submit on BlackBoard

1) For the following RLC circuit with C6 = 0(remove C6)

- Specify the dynamics for the system (write N coupled differential equations)
- Express these dynamics in state-space form
- Determine the transfer function from Vin to Y



Option #1: Without C6

$$V_1 = 0.5sI_1 = V_{in} - 10I_1 - V_4$$

$$V_2 = 0.2sI_2 = V_{in} - 5I_2 - V_4$$

$$V_3 = 0.25sI_3 = V_4 - 15I_3$$

$$I_4 = 0.1sV_4 = I_1 + I_2 - I_3 + \left(\frac{V_5 - V_4}{50}\right)$$

$$I_5 = 0.5sV_5 = \left(\frac{V_4 - V_5}{50}\right)$$

$$Y = \left(\frac{4}{5}\right)V_4 + \left(\frac{1}{5}\right)V_5$$

Group terms and simplify

$$sI_1 = 2V_{in} - 20I_1 - 2V_4$$

$$sI_2 = 5V_{in} - 25I_2 - 5V_4$$

$$sI_3 = 4V_4 - 60I_3$$

$$sV_4 = 10I_1 + 10I_2 - 10I_3 + 0.2V_5 - 0.2V_4$$

$$sV_5 = 0.04V_4 - 0.04V_5$$

Place in matrix form

$$\begin{bmatrix} sI_1 \\ sI_2 \\ sI_3 \\ sV_4 \\ sV_5 \end{bmatrix} = \begin{bmatrix} -20 & 0 & 0 & -2 & 0 \\ 0 & -25 & 0 & -5 & 0 \\ 0 & 0 & -60 & 4 & 0 \\ 10 & 10 & -10 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0.04 & -0.04 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \end{bmatrix} + [0]V_{in}$$

Find the transfer funciton using Matlab

```
>> A = [-20,0,0,-2,0;0,-25,0,-5,0;0,0,-60,4,0];
>> A = [A;10,10,-10,-0.2,0.2;0,0,0,0.04,-0.04]

  -20.0000         0         0   -2.0000         0
         0  -25.0000         0   -5.0000         0
         0         0  -60.0000    4.0000         0
   10.0000   10.0000  -10.0000   -0.2000    0.2000
         0         0         0    0.0400   -0.0400

>> B = [2;5;0;0;0];
>> C = [0,0,0,0.8,0.2];
>> D = 0;
>> G5 = ss(A,B,C,D);
>> zpk(G5)


           56 (s+21.43) (s+60) (s+0.05)
---------------------------------------------------
(s+59.35) (s+23.22) (s+17.94) (s+4.694) (s+0.03791)

>>
```

Option #2: With C6 = 0.001. Write the dynamics

$$V_1 = 0.5sI_1 = V_{in} - 10I_1 - V_4$$

$$V_2 = 0.2sI_2 = V_{in} - 5I_2 - V_4$$

$$V_3 = 0.25sI_3 = V_4 - 15I_3$$

$$I_4 = 0.1sV_4 = I_1 + I_2 - I_3 - \left(\frac{V_4 - V_6}{10}\right)$$

$$I_5 = 0.5sV_5 = \left(\frac{V_6 - V_5}{40}\right)$$

$$I_6 = 0.001sV_6 = \left(\frac{V_4 - V_6}{10}\right) - \left(\frac{V_6 - V_5}{40}\right)$$

Group terms and solve for the highest derivative

$$sI_1 = 2V_{in} - 20I_1 - 2V_4$$

$$sI_2 = 5V_{in} - 25I_2 - 5V_4$$

$$sI_3 = 4V_4 - 60I_3$$

$$sV_4 = 10I_1 + 10I_2 - 10I_3 - V_4 + V_6$$

$$sV_5 = 0.05V_6 - 0.05V_5$$

$$sV_6 = 100V_4 - 125V_6 + 25V_5$$

Place in state-space form (matrix form)

$$s\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} -20 & 0 & 0 & -2 & 0 & 0 \\ 0 & -25 & 0 & -5 & 0 & 0 \\ 0 & 0 & -60 & 4 & 0 & 0 \\ 10 & 10 & -10 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.05 & 0.05 \\ 0 & 0 & 0 & 100 & 25 & -125 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = V_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} + [0]V_{in}$$

Find the transfer funciton using Matlab

```
>> a1 = [-20,0,0,-2,0,0];
>> a2 = [0,-25,0,-5,0,0];
>> a3 = [0,0,-60,4,0,0];
>> a4 = [10,10,-10,-1,0,1];
>> a5 = [0,0,0,0,-0.05,0.05];
>> a6 = [0,0,0,100,25,-125];
>> A = [a1;a2;a3;a4;a5;a6]

  -20.0000         0         0   -2.0000         0         0
         0  -25.0000         0   -5.0000         0         0
         0         0  -60.0000    4.0000         0         0
   10.0000   10.0000  -10.0000   -1.0000         0    1.0000
         0         0         0         0   -0.0500    0.0500
         0         0         0  100.0000   25.0000 -125.0000

>> B = [2;5;0;0;0;0]

     2
     5
     0
     0
     0
     0


>> C = [0,0,0,0,0,1];
>> D = 0;
>> G6 = ss(A,B,C,D);

>> zpk(G6)

               7000 (s+21.43) (s+60) (s+0.05)
-------------------------------------------------------------
(s+125.8) (s+59.35) (s+23.23) (s+17.97) (s+4.655) (s+0.03791)
```

which is about the same as the 5th-order model, but with a fast poles added (C = 0.001)

```
>> zpk(G5)

           56 (s+21.43) (s+60) (s+0.05)
---------------------------------------------------
(s+59.35) (s+23.22) (s+17.94) (s+4.694) (s+0.03791)
```

2) For the transfer function from V0 to V1

- Determine a 1st or 2nd-order approximation for this trasfer function
- Plot the step response of the actual 4th-order system and its approximation

```
            56 (s+21.43) (s+60) (s+0.05)
---------------------------------------------------
(s+59.35) (s+23.22) (s+17.94) (s+4.694) (s+0.03791)
```

Keep the dominant pole

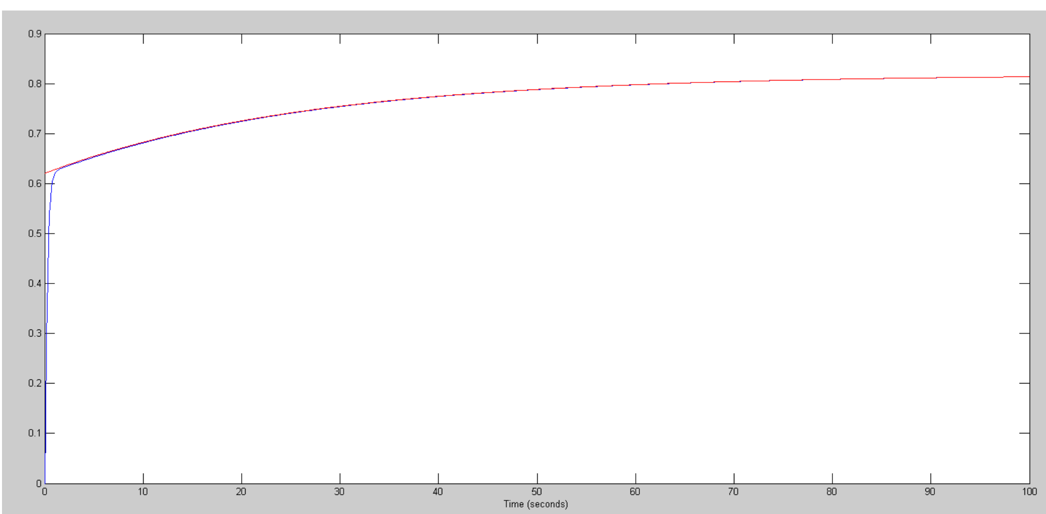
- s = -0.03791

Also keep the zero since it's close to that pole

- s = -0.05;

Match the DC gain

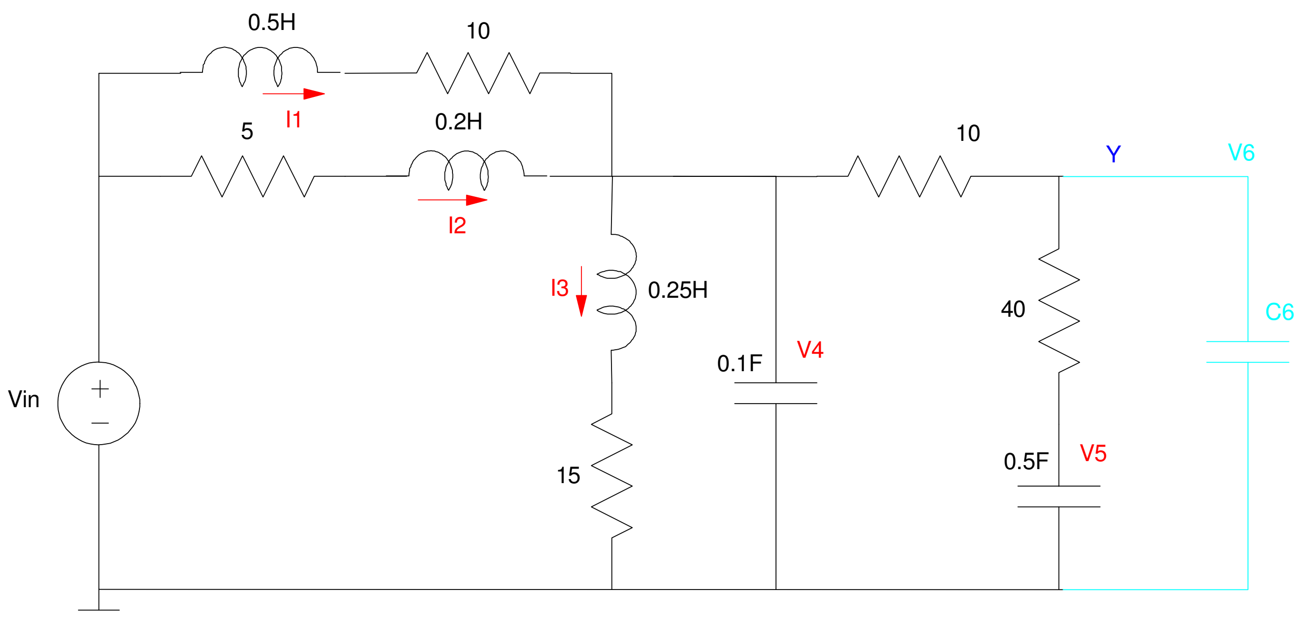
```
>> DC = evalfr(G5,0)

DC =    0.8182

>> G1 = zpk(-0.05,-0.03791,1)
>> k = evalfr(G5,0) / evalfr(G1,0)

k =    0.6203

>> G1 = zpk(-0.05,-0.03791,0.6203)

0.6203 (s+0.05)
---------------
  (s+0.03791)

>> t = [0:0.1:100]';
>> y5 = step(G5,t);
>> y1 = step(G1,t);
>> plot(t,y5,'b',t,y1,'r');
>> xlabel('Time (seconds)');
>>
```



5th-Order System (blue) & 1st-order approximation (red)

3) For this circuit

- What initial condition will the energy in the system decay as slowly as possible?
- What initial condition will the energy in the system decay as fast as possible?



This is an eigenvalue / eigenvector problem

```
>> [M,V] = eig(A6)

M =
```

	ignore	fast				slow
I1	-0.0001	0.0079	0.1920	0.5372	-0.0980	0.0051
I2	-0.0004	0.0226	-0.8790	0.3882	-0.1847	0.0101
I3	0.0005	0.9590	0.0338	-0.0520	0.0543	-0.0034
V4	-0.0079	0.1551	0.3105	-0.5460	0.7516	-0.0507
V5	-0.0004	-0.0002	-0.0007	0.0014	-0.0068	-0.9707
V6	1.0000	0.2361	0.3049	-0.5098	0.6232	-0.2347

```
V =

 -125.8030  -59.3532  -23.2340  -17.9674   -4.6545   -0.0379
```

The fastest eigenvector is the 0.001F capacitor we added. Ignore this one since we addded it to the circuit to make the equations easier to write

The red eigenvector is the fast mode, decaying as exp(-59.35t)

- put most of the initial energy into I3

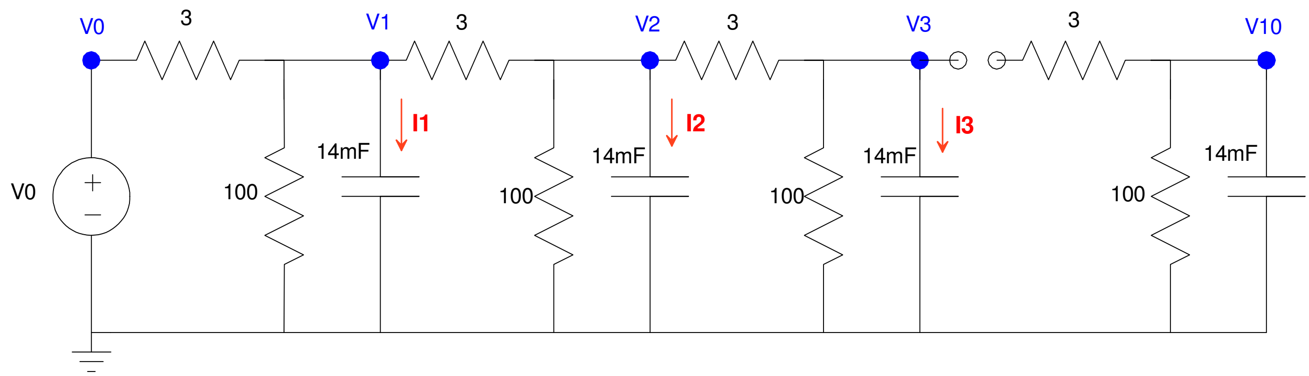
The blue eigenvector is the slow mode, decayig as exp(-0.0379t)

- put most of the initial energy into V5

Problem 4-7: 10-Stage RC Filter.

4) For the following 10-stage RC circuit

- Specify the dynamics for the system (write N coupled differential equations)
  - note: Nodes 1..9 have the same form. Just write the node equation for node 1 and node 10.
- Express these dynamics in state-space form
- Determine the transfer function from V0 to V10



Start with node V1

$$I_1 = 0.014 s V_1 = \left(\frac{V_0 - V_1}{3}\right) + \left(\frac{V_2 - V_1}{3}\right) - \left(\frac{V_1}{100}\right)$$

$$sV_1 = 23.81 V_0 - 48.33 V_1 + 23.81 V_2$$

By symmetry, the same holds for nodes 2..9

$$sV_2 = 23.81 V_1 - 48.33 V_2 + 23.81 V_3$$

$$sV_3 = 23.81 V_2 - 48.33 V_3 + 23.81 V_4$$

$$\vdots$$

$$sV_9 = 23.81 V_8 - 48.33 V_9 + 23.81 V_{10}$$

Node #10 is a little difference since there is only one 3-Ohm resistor connected to it

$$I_{10} = 0.014 s V_{10} = \left(\frac{V_9 - V_{10}}{3}\right) + \left(\frac{V_{10}}{100}\right)$$

$$sV_{10} = 23.81 V_9 - 24.52 V_{10}$$

Place in matrix form

$$
\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \\ sV_4 \\ sV_5 \\ sV_6 \\ sV_7 \\ sV_8 \\ sV_9 \\ sV_{10} \end{bmatrix}
=
\begin{bmatrix}
-48.33 & 23.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23.81 & -48.33 & 23.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 23.81 & -48.33 & 23.81 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 23.81 & -48.33 & 23.81 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 23.81 & -48.33 & 23.81 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 23.81 & -48.33 & 23.81 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 23.81 & -48.33 & 23.81 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 23.81 & -48.33 & 23.81 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 23.81 & -48.33 & 23.81 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23.81 & -24.52
\end{bmatrix}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix}
+
\begin{bmatrix} 23.81 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}
$$

Place into Matlab and find the transfer funciton

```
>> A = zeros(10,10);
>> for i=1:9
A(i,i) = -48.33;
A(i,i+1) = 23.81;
A(i+1,i) = 23.81;
end
>> A(10,10) = -24.52

A =

  -48.3300   23.8100         0         0         0         0         0         0         0         0
   23.8100  -48.3300   23.8100         0         0         0         0         0         0         0
         0   23.8100  -48.3300   23.8100         0         0         0         0         0         0
         0         0   23.8100  -48.3300   23.8100         0         0         0         0         0
         0         0         0   23.8100  -48.3300   23.8100         0         0         0         0
         0         0         0         0   23.8100  -48.3300   23.8100         0         0         0
         0         0         0         0         0   23.8100  -48.3300   23.8100         0         0
         0         0         0         0         0         0   23.8100  -48.3300   23.8100         0
         0         0         0         0         0         0         0   23.8100  -48.3300   23.8100
         0         0         0         0         0         0         0         0   23.8100  -24.5200

>> B = [23.81;0;0;0;0;0;0;0;0;0];
>> C = [0,0,0,0,0,0,0,0,0,0,1];
>> D = 0;
>> G10 = ss(A,B,C,D);
??? Error using ==> ss.ss>ss.ss at 345
The values of the "a" and "c" properties must be matrices with the same number of
columns.

>> C = [0,0,0,0,0,0,0,0,0,1];
>> D = 0;
>> G10 = ss(A,B,C,D);
>> zpk(G10)

                                      58559040760278.24
------------------------------------------------------------------------------------------------
(s+93.83) (s+87.68) (s+78.02) (s+65.73) (s+51.89) (s+37.73) (s+24.52) (s+13.42) (s+5.426) (s+1.242)

>>
```

5) For the transfer function for problem #4

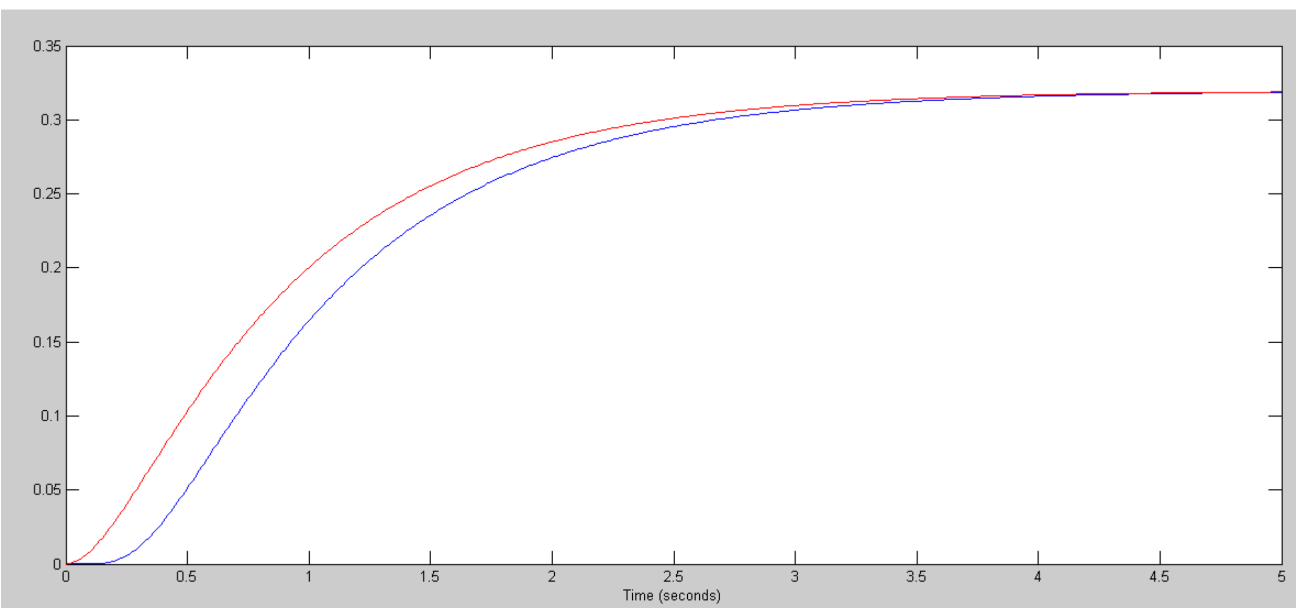
- Determine a 2nd-order approximation for this trasfer function
- Plot the step response of the actual 10th-order system and its 2nd-order approximation

```
                                    58559040760278.24
---------------------------------------------------------------------------------------------
(s+93.83) (s+87.68) (s+78.02) (s+65.73) (s+51.89) (s+37.73) (s+24.52) (s+13.42) (s+5.426) (s+1.242)
```

Keep the two most dominant poles (in red)

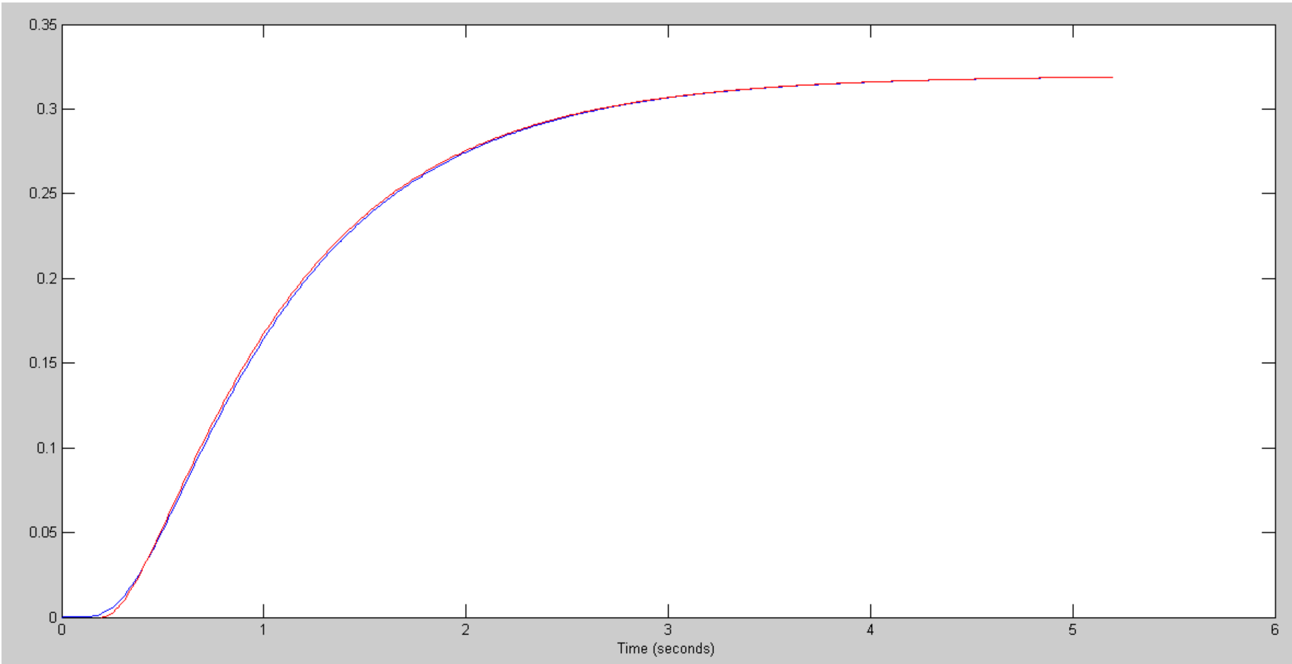
Match the DC gain

```
>> DC = evalfr(G10,0)

DC =    0.3197

>> G2 = zpk([],[-1.242,-5.426],1);
>> k = evalfr(G10,0) / evalfr(G2,0)

k =    2.1543

>> G2 = zpk([],[-1.242,-5.426],2.1543)

      2.1543
-------------------
(s+1.242) (s+5.426)

>> t = [0:0.01:5]';
>> y10 = step(G10,t);
>> y2 = step(G2,t);
>> plot(t,y10,'b',t,y2,'r');
>> xlabel('Time (seconds)');
>>
```



Sidelight - it is a little more accurate if you add a delay

```
>> plot(t,y10,'b',t+0.2,y2,'r');
>> xlabel('Time (seconds)');
```



so a better model (that doesn't fit into state-space form)

$$G(s) \approx \left(\frac{2.1543}{(s+1.242)(s+5.426)}\right) e^{-0.2s}$$

6) For the circuit for problem #4

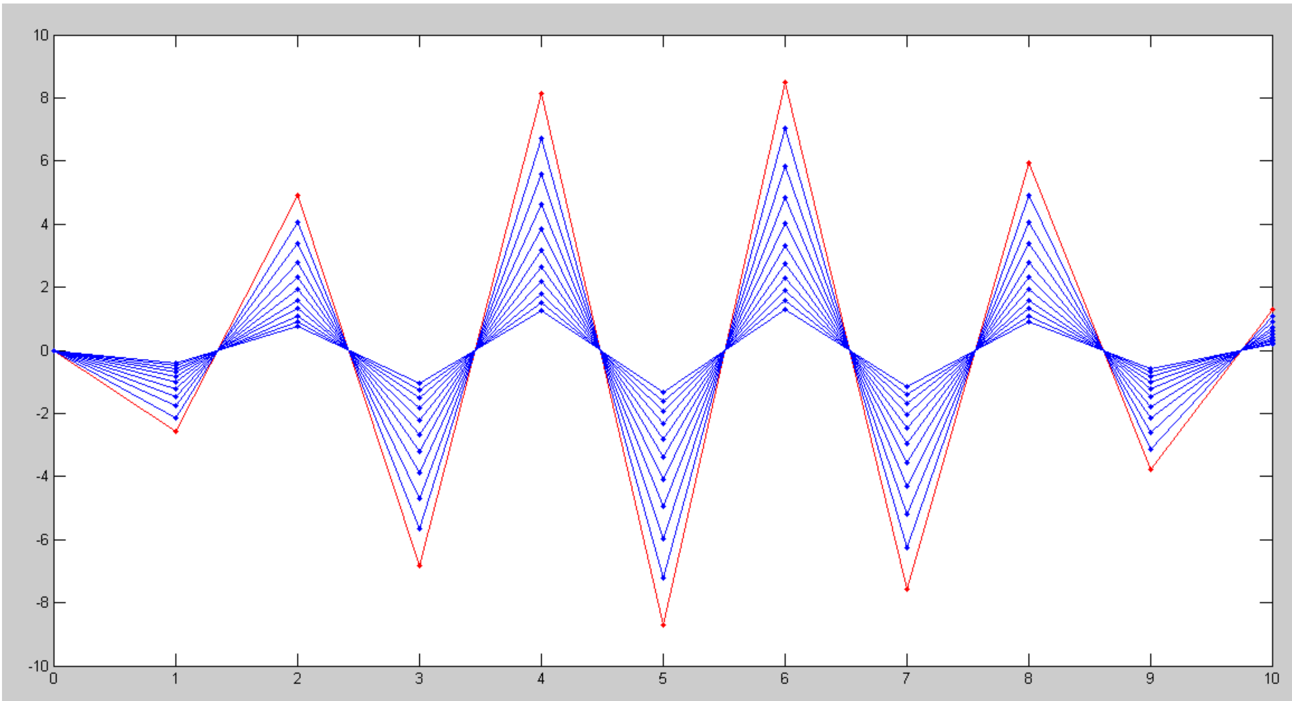
- What initial condition will decay as slowly as possible?
- What initial condition will decay as fast as possible?

This is an eigenvector / eigenvalue problem

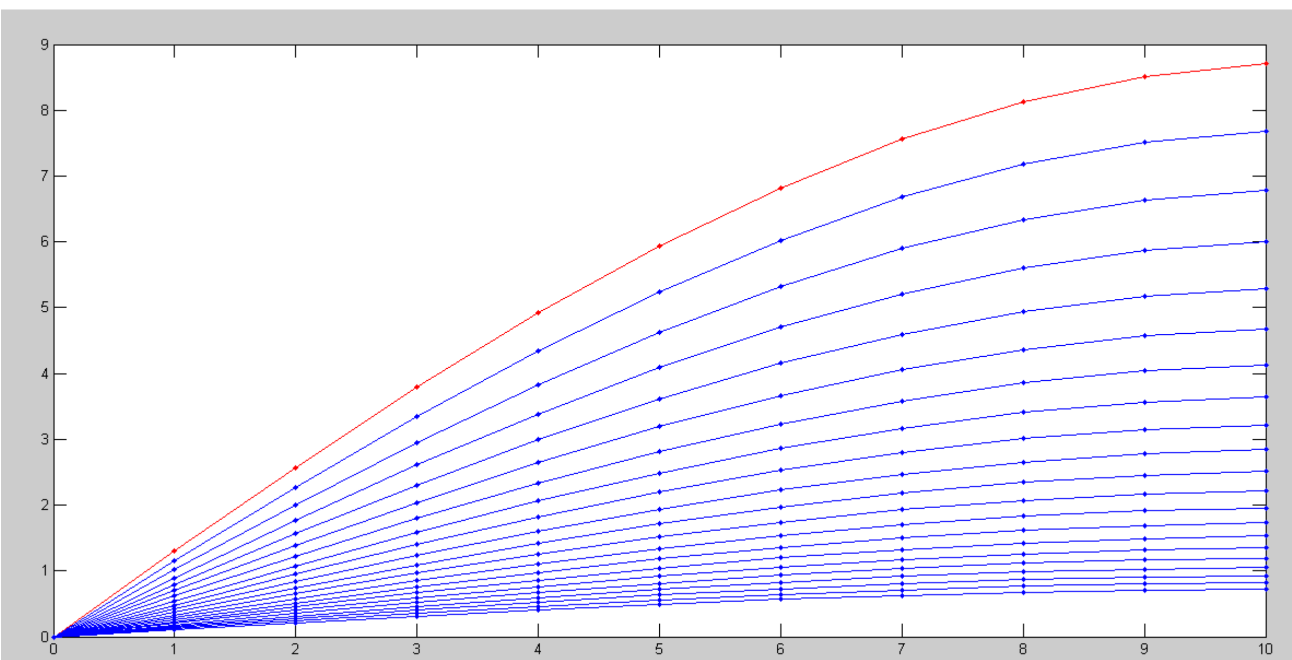
```
>> [M,V] = eig(A)

M = (eigenvectors)

     fast                                                                                     slow
   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
    0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412    0.0000    0.3412   -0.4255    0.1894
    0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
   -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255   -0.0000   -0.4255   -0.1894    0.3412
   -0.3780    0.3780   -0.0000    0.3780   -0.3780   -0.0000    0.3780   -0.3780    0.0000    0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
    0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352

V (eigenvalues)

  -93.8344  -87.6755  -78.0206  -65.7275  -51.8886  -37.7336  -24.5200  -13.4221   -5.4259   -1.2419
```

7) Modify the program *heat.m* to match the dynamics you calculated for this problem.

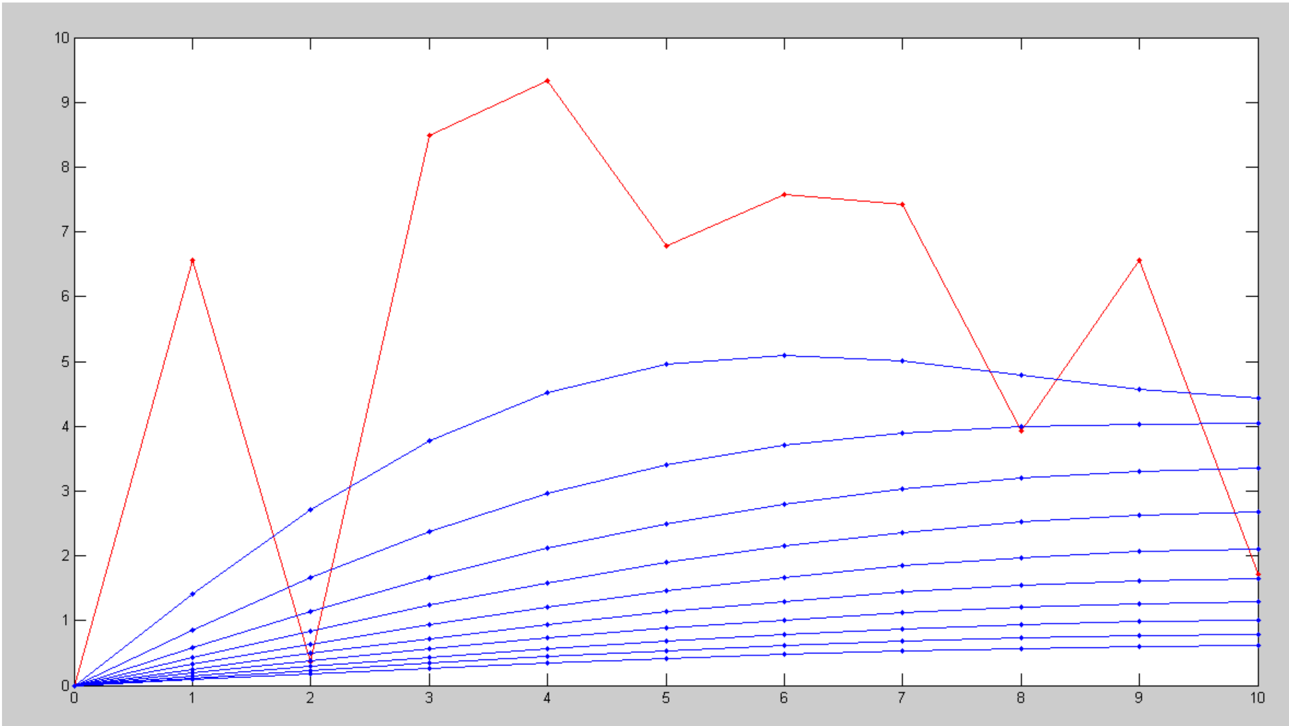
- Give the program listing
- Give the response for Vin = 0 and the initial conditions being
  - The slowest eigenvector
  - The fastest eigenvector
  - A random set of voltages



Fast Mode: Decay as exp(-93.8t)



Slow Mode: Decay as exp(-1.24t)



Random Initial Condition: Fast modes decay quickly, leaving the slow mode

Program Listing

```
% 10-stage RC Filter

%V = M(:,10) * 20;
%V = M(:,1) * 20;
V = 10*rand(10,1)

dV = zeros(10,1);
V0 = 0;
dt = 0.002;
t = 0;
i = 0;

y = [];

plot([0:10], [V0;V], 'r.-');
hold on

while(t < 2)
   dV(1)  = 23.81*V0   - 48.33*V(1) + 23.81*V(2);
   dV(2)  = 23.81*V(1) - 48.33*V(2) + 23.81*V(3);
   dV(3)  = 23.81*V(2) - 48.33*V(3) + 23.81*V(4);
   dV(4)  = 23.81*V(3) - 48.33*V(4) + 23.81*V(5);
   dV(5)  = 23.81*V(4) - 48.33*V(5) + 23.81*V(6);
   dV(6)  = 23.81*V(5) - 48.33*V(6) + 23.81*V(7);
   dV(7)  = 23.81*V(6) - 48.33*V(7) + 23.81*V(8);
   dV(8)  = 23.81*V(7) - 48.33*V(8) + 23.81*V(9);
   dV(9)  = 23.81*V(8) - 48.33*V(9) + 23.81*V(10);
   dV(10) = 23.81*V(9) - 24.52*V(10);

   V = V +  dV*dt;
   t = t + dt;

   y = [y ; V'];
   i = mod(i + 1, 100);
   if(i == 0)
      plot([0:10], [V0;V], '.-');
      pause(0.01);
      ylim([0,10]);
   end

   end
```