ECE 463/663 - Homework #3

Canonical Forms, Similarity Transforms, LaGrangian Dynamics, Block Diagrams
Due Monday, January 29th
Please submit as a hard copy or submit on BlackBoard

Canonical Forms

Problem 1-3) For the system

$$Y = \left(\frac{30(s+2)(s+3)}{(s+1)(s+5)(s+10)} \right) U$$

1) Express this system in controller canonical form. (Give the A, B, C, D matrices)

Multiply out

```
>> poly([-2,-3])*30

ans =    30   150   180

>> poly([-1,-5,-10])

ans =     1    16    65    50

>> A = [0,1,0;0,0,1;-50,-65,-16]

     0     1     0
     0     0     1
   -50   -65   -16

>> B = [0;0;1]

     0
     0
     1

>> C = [180,150,30]

   180   150    30

>> D = 0

   0

>> G = ss(A,B,C,D);
>> zpk(G)

  30 (s+3) (s+2)
 ------------------
 (s+10) (s+5) (s+1)
```

Check: (A, B, C, D) give the correct transfer funciton

2) Express this system in cascade form

$$Y = \left(\frac{30(s+2)(s+3)}{(s+1)(s+5)(s+10)} \right) U$$

Express this as

$$Y = \left(\left(\frac{a}{(s+1)(s+5)(s+10)} \right) + \left(\frac{b}{(s+5)(s+10)} \right) + \left(\frac{c}{(s+10)} \right) \right) U$$

Place over a common denominator and the numberator is

$$a + b(s+1) + c(s+1)(s+5) = 30(s+2)(s+3)$$

$$cs^2 + (b+6c)s + (a+b+5c) = 30s^2 + 150s + 180$$

Solving

c = 30

b = -30

a = 60

```
>> A = [-10,0,0;1,-5,0;0,1,-1]

   -10     0     0
     1    -5     0
     0     1    -1

>> B = [1;0;0]

     1
     0
     0

>> C = [30,-30,60]

C =     30   -30    60

>> D = 0

D =      0

>> G = ss(A,B,C,D);
>> zpk(G)


  30 (s+3) (s+2)
------------------
(s+1) (s+5) (s+10)
```

3) Express this system in Jordan (diagonal) form

Do a partial fraction expansion

$$Y = \left(\frac{30(s+2)(s+3)}{(s+1)(s+5)(s+10)} \right) U$$

$$Y = \left(\left(\frac{1.667}{s+1} \right) + \left(\frac{-9}{s+5} \right) + \left(\frac{37.333}{s+10} \right) \right) U$$

```
>> A = diag([-1,-5,-10])

    -1     0     0
     0    -5     0
     0     0   -10

>> B = [1.667;-9;37.333]

    1.6670
   -9.0000
   37.3330

>> C = [1,1,1]

C =     1     1     1

>> D = 0

D = 0

>> G = ss(A,B,C,D);
>> zpk(G)

   30 (s+3) (s+2)
------------------
(s+10) (s+5) (s+1)

>>
```

4) Assume a system's dynamics are

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \\ sV_4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} V_0$$

$$Y = V_1 - V_2$$

Express these dynamic with the change in variable

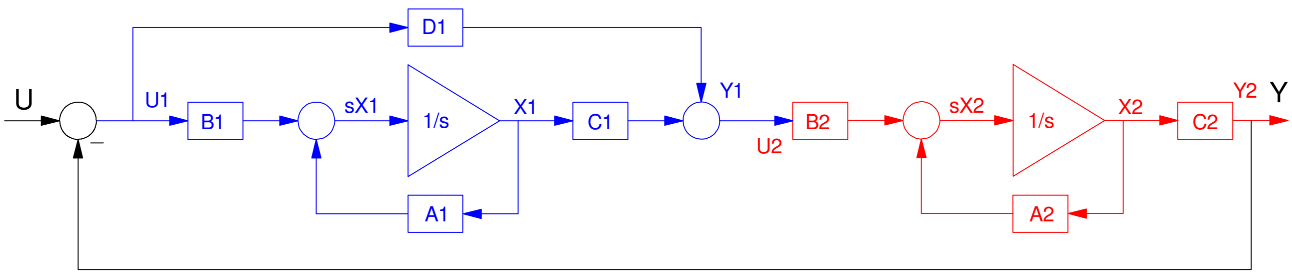
$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 + V_2 \\ V_1 + V_2 + V_3 \\ V_1 + V_2 + V_3 + V_4 \end{bmatrix}$$

```
>> A = [-2,1,0,0;1,-3,1,0;0,1,-4,1;0,0,1,-5]

    -2     1     0     0
     1    -3     1     0
     0     1    -4     1
     0     0     1    -5

>> B = [1;1;2;2]

     1
     1
     2
     2

>> C = [1,-1,0,0]

C =     1    -1     0     0

>> D = 0;
>> Ti = [1,0,0,0;1,1,0,0;1,1,1,0;1,1,1,1]

     1     0     0     0
     1     1     0     0
     1     1     1     0
     1     1     1     1

>> T = inv(Ti)

     1     0     0     0
    -1     1     0     0
     0    -1     1     0
     0     0    -1     1
```

```
>> Az = inv(T)*A*T

    -3     1     0     0
     1    -3     1     0
     0     2    -4     1
     0     1     2    -4

>> Bz = inv(T)*B

     1
     2
     4
     6

>> Cz = C*T

Cz =     2    -1     0     0

>> Dz = D

Dz =     0

>>
```

Block Diagrams

5) Determine the state-space model the following system:



Writing the equations for sX

$$sX_1 = B_1U + A_1X_1 - B_1C_2X_2$$

$$sX_2 = A_2X_2 + B_2C_1X_1 + B_2D_1U - B_2D_1C_2X_2$$

Equation for Y

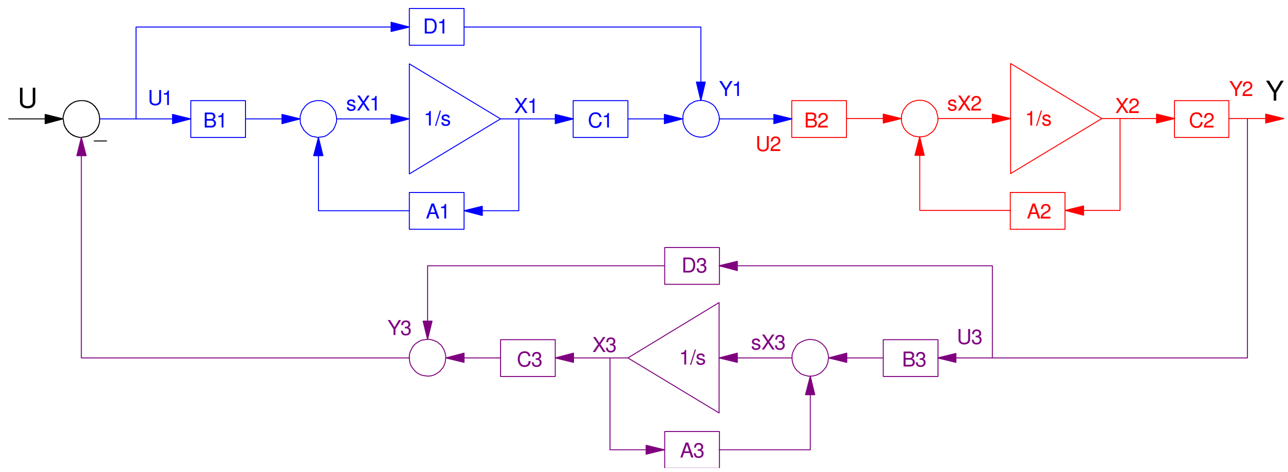
$$Y = C_2X_2$$

In matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 - B_2D_1C_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [0]U$$

6) Determine the state-space model for the following system:



Write the equations at the input to the integrators (sX)

$$sX_1 = A_1X_1 + B_1U - B_1C_3X_3 - B_1D_3C_2X_2$$

$$sX_2 = A_2X_2 + B_2C_1X_1 + B_2D_1U - B_2D_1C_3X_3 - B_2D_1D_3C_2X_2$$

$$sX_3 = A_3X_3 + B_3C_2X_2$$

and the equation for Y

$$Y = C_2X_2$$

Place in matrix form

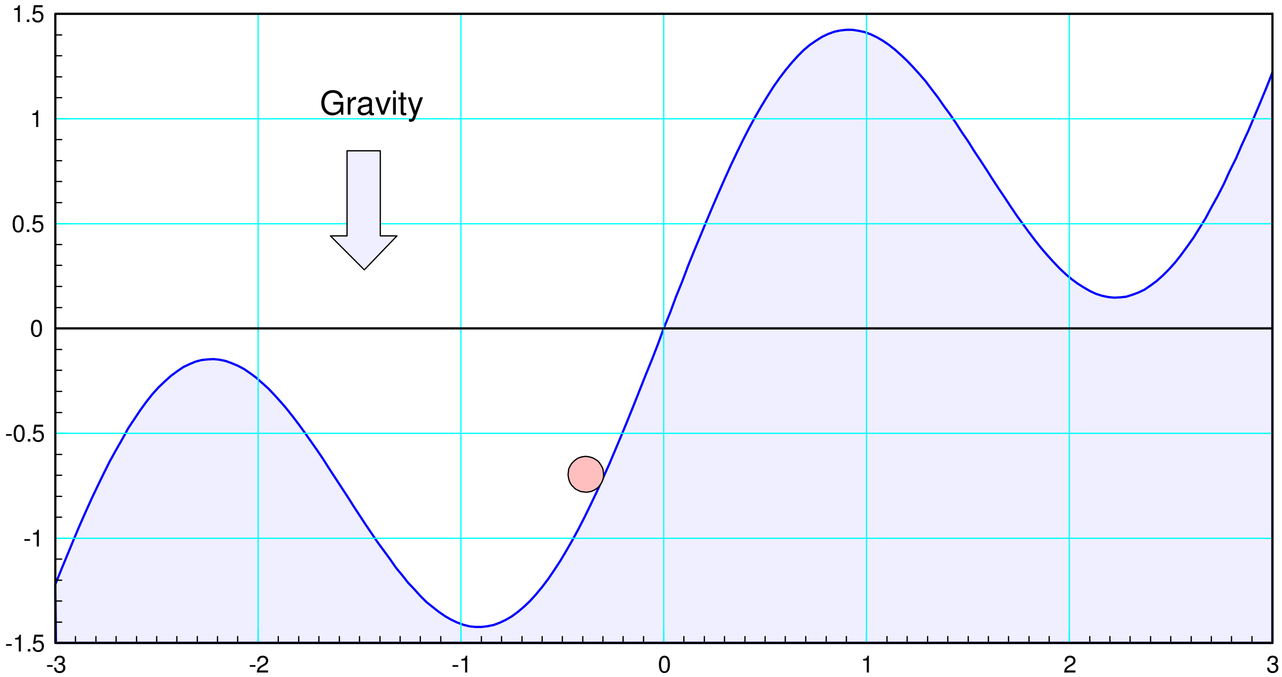
$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1D_3C_2 & -B_1C_3 \\ B_2C_1 & A_2 - B_2D_1D_3C_2 & -B_2D_1C_3 \\ 0 & B_3C_2 & A_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + [0]U$$

LaGrangian Dynamics

A 1kg ball is rolling on a surface defined by:

$$y = 0.5x + \sin(2x)$$



7) Determine the kinetic and potential energy of this ball as a function of x: Gravity is in the -y direction.

PE:

$$PE = mgh = 0.5gx + g\sin(2x)$$

KE: Assuming a solid sphere with m = 1 kg

$$y = 0.5x + \sin(2x)$$

$$\dot{y} = 0.5\dot{x} + 2\cos(2x)\dot{x}$$

$$KE = 0.7(\dot{x}^2 + \dot{y}^2)$$

$$KE = 0.7\left(\dot{x}^2 + (0.5\dot{x} + 2\cos(2x)\dot{x})^2\right)$$

$$= 0.7(\dot{x}^2 + 0.25\dot{x}^2 + 2\cos(2x)\dot{x}^2 + 4\cos^2(2x)\dot{x}^2)$$

$$= 0.7\left(1.25\dot{x}^2 + 2\cos(2x)\dot{x}^2 + 4\left(\frac{1+\cos(4x)}{2}\right)\dot{x}^2\right)$$

$$= 2.275\dot{x}^2 + 1.4\cos(2x)\dot{x}^2 + 1.4\cos(4x)\dot{x}^2$$

8) Determine the dynamics for this ball as it rolls on this surface

Set up the LaGrangian

$$L = KE - PE$$

$$L = (2.275\dot{x}^2 + 1.4\cos(2x)\dot{x}^2 + 1.4\cos(4x)\dot{x}^2) - g(0.5x + \sin(2x))$$

Take partials

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}(4.55\dot{x} + 2.8\cos(2x)\dot{x} + 2.8\cos(4x)\dot{x})$$
$$-(-2.8\sin(2x)\dot{x}^2 - 5.6\sin(4x)\dot{x}^2 - 0.5g - 2g\cos(2x))$$

$$F = (4.55 + 2.8\cos(2x) + 2.8\cos(4x))\ddot{x}$$
$$-5.6\sin(2x)\dot{x}^2 - 11.2\sin(4x)\dot{x}^2$$
$$+2.8\sin(2x)\dot{x}^2 + 5.6\sin(4x)\dot{x}^2 + 0.5g + 2g\cos(2x)$$

$$F = (4.55 + 2.8\cos(2x) + 2.8\cos(4x))\ddot{x}$$
$$-2.8\sin(2x)\dot{x}^2 - 5.6\sin(4x)\dot{x}^2 + 0.5g + 2g\cos(2x)$$

If F = 0, then the acceleration is

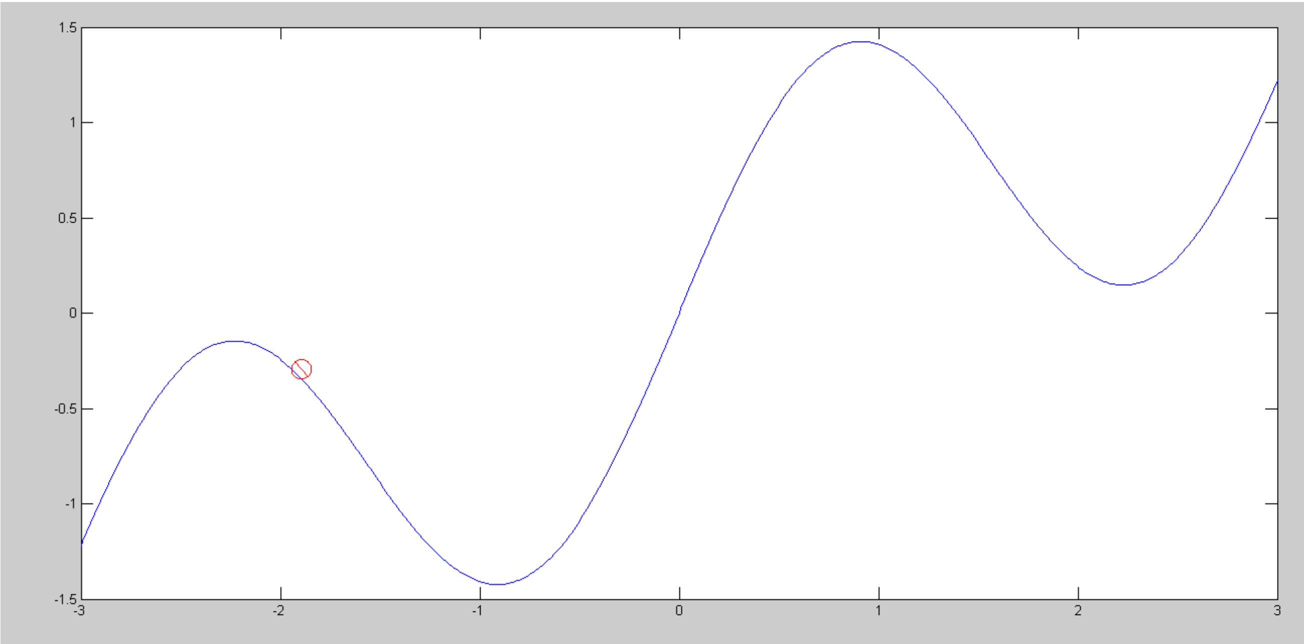
$$\ddot{x} = \left(\frac{2.8\sin(2x)\dot{x}^2 + 5.6\sin(4x)\dot{x}^2 - 0.5g - 2g\cos(2x)}{4.55 + 2.8\cos(2x) + 2.8\cos(4x)}\right)$$

another way to write this using trig identities

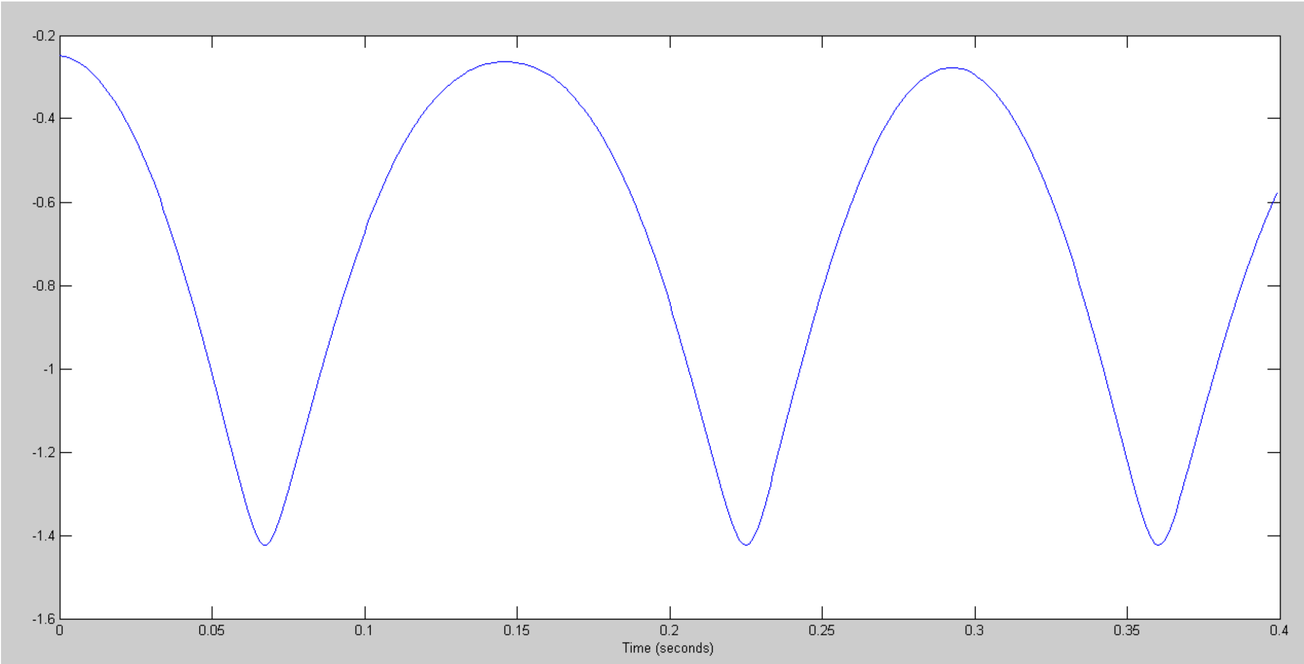
$$\ddot{x} = \left(\frac{2.8\sin(2x)\dot{x}^2 + 2.8\sin^2(4x)\dot{x}^2 - 0.5g - 2g\cos(2x)}{1.75 + 2.8\cos(2x) + 5.6\cos^2(2x)}\right)$$

Simulation Results:

- x(0) = -0.1
- x'(0) = 0



Ball Rolling on a Surface



Y positon of the ball vs. time. Max height is the same going left and right

Code

```
x = -0.10;
dx = 0;
dt = 0.001;
t = 0;
g = 9.8;
L = 0;
q = 0;  % angle of ball for animation
r = 0.05; % radius of ball for animation

Yd = []; % display matrix
n = 0;

while(t < 4)

% compute the acceleration
   num = 2.8*sin(2*x)*dx*dx + 5.6*sin(4*x)*dx*dx - (0.5 + 2*cos(2*x))*g;
   den = 4.55 + 2.8*cos(2*x) + 2.8*cos(4*x);
   ddx = num / den;

% compute how much the ball is rotating
   dydx = 0.5 + 2*cos(2*x);
   dq = sqrt(1 + dydx^2) / r * dx;

% integration
   dx = dx + ddx*dt;
   x = x + dx*dt;
   q = q + dq*dt;
   t = t + dt;

% display the ball
   y = 0.5*x + sin(2*x);

   x1 = [-3:0.01:3]';
   y1 = 0.5*x1 + sin(2*x1);

 % draw the ball
   i = [0:0.01:1]' * 2 * pi;
   xb = r*cos(i) + x;
   yb = r*sin(i) + y + r;

 % line through the ball
   Q = [0, pi] - q;
   xb1 = 0.05*cos(Q) + x;
   yb1 = 0.05*sin(Q) + y + r;

% plot the ball every 10th pass
   n = mod(n+1,10);
   if(n == 0)
         plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');
         pause(0.01);
         ylim([0,1]);
         xlim([-2,2]);
         Yd = [Yd ; y];
         end

   end

t = [0:length(Yd)-1]' * dt;
plot(t,Yd);
xlabel('Time (seconds)');
```