ECE 463/663 - Homework #9

Calculus of Variations. Ricatti Equation. Due Wednesday, April 3rd

Please submit as a hard copy, email to jacob.glower@ndsu.edu, or submit on BlackBoard

Soap Film

1) Calculate the shape of a soap film connecting two rings around the X axis:
   - Y(0) = 6
   - Y(4) = 9

From the lecture notes, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

Plugging in the endpoint constraints:

$$6 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

$$9 = a \cdot \cosh\left(\frac{4-b}{a}\right)$$

Solve in Matlab: Create a function to minimize

```
function [J] = Soap(z)
   a = z(1);
   b = z(2);
   e1 = a*cosh((0-b)/a) - 6;
   e2 = a*cosh((4-b)/a) - 9;
   J = e1^2 + e2^2;
   end
```

Solve using *fminsearch()*

```
>> [Z,e] = fminsearch('Soap',[1,2])
         a         b
Z =    0.6380    1.8702
e =  4.8520e-007
```

So, one solution is

$$y = 0.6380 \cdot \cosh\left(\frac{x-1.8702}{0.6380}\right)$$

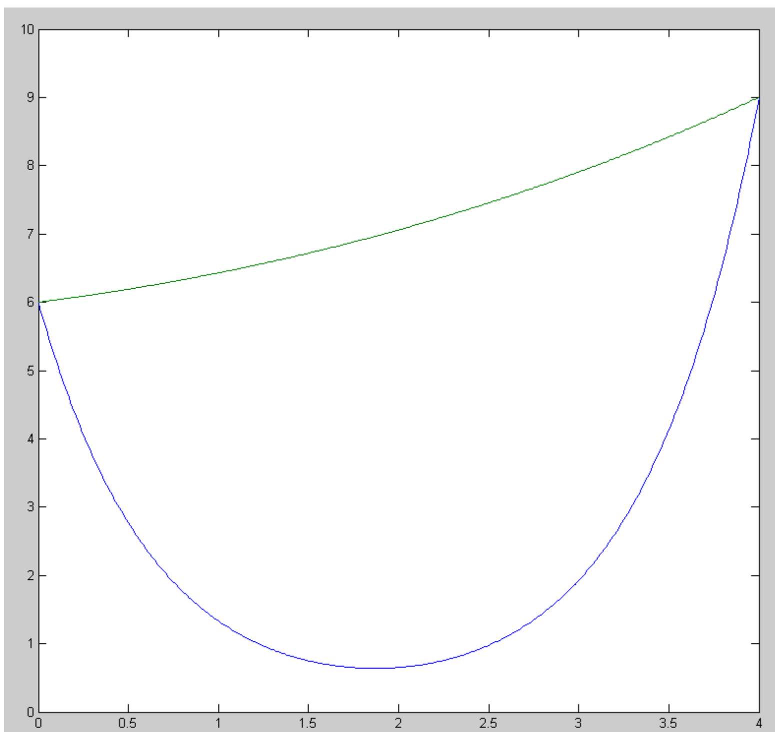
A second solution is

```
>> [Z,e] = fminsearch('Soap',[51,2])
        a          b
Z =    5.6887   -1.8736
e =  8.6887e-010
```

$$y = 5.6887 \cdot \cosh\left(\frac{x+1.8736}{5.6887}\right)$$

Plotting these

```
>> [Z,e] = fminsearch('Soap',[1,2])

Z =     0.6380    1.8702

e =   4.8520e-007

>> a = Z(1);
>> b = Z(2);
>> y = a * cosh( (x-b)/a );
>> [Z,e] = fminsearch('Soap',[51,2])

Z =     5.6887   -1.8736

e =   8.6887e-010

>> a = Z(1);
>> b = Z(2);
>> y2 = a * cosh( (x-b)/a );
>> plot(x,y,x,y2)
```



Two solutions for the soap-film problem (?)

2) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 6
- Y(2) = free

Again, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

The left endpoint must satisfy

$$6 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

The right endpoint must satisfy $F_{y'} = 0$

$$y' = \sinh\left(\frac{x-b}{a}\right) = 0$$

$$\sinh\left(\frac{2-b}{a}\right) = 0$$

Solving in Matlab using fminsearch:

```
function [J] = Soap(z)
   a = z(1);
   b = z(2);
   e1 = a*cosh((0-b)/a) - 6;
   e2 = sinh((2-b)/a);
   J = e1^2 + e2^2;
   end
```

In the command window:

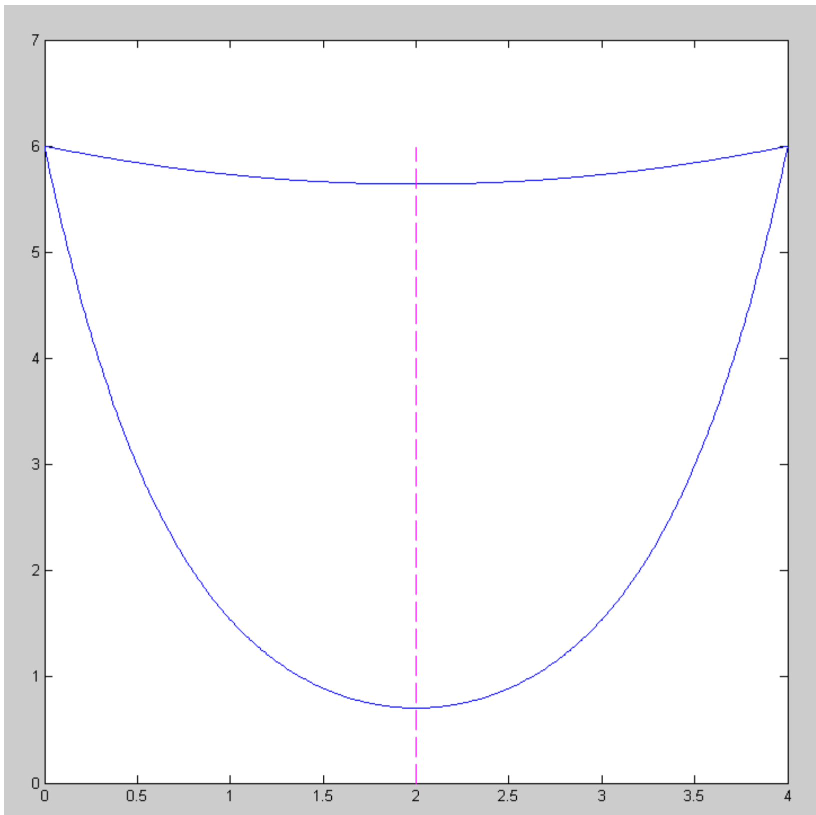
```
>> [Z,e] = fminsearch('Soap',[51,2])

         a         b
Z =    5.6418    2.0000

e =   2.0802e-011
```

$$y = 5.6418 \cosh\left(\frac{x-2}{5.6418}\right)$$

A second solution is:

```
>> [Z,e] = fminsearch('Soap',[1,2])

         a         b
Z =    0.7073    2.0000

e =   2.4626e-009
```

```
>> a = Z(1);
>> b = Z(2);
>> y = a * cosh( (x-b)/a );
>> hold on
>> a = Z(1);
>> b = Z(2);
>> x = [0:0.01:4]';
>> y = a * cosh( (x-b)/a );
>> plot(x,y,[2,2],[0,6],'m--')
```



Two solutions for the soap-file problem (?)

Hanging Chain

3) Calculate the shape of a hanging chain subject to the following constraints

- Length of chain = 11 meters
- Left Endpoint: (0,7)
- Right Endpoint: (10,8)

From the lecture notes, a hanging chain

- Minimizes the potential energy,
- With the constraint that the total lenngth is 12 meters

The corresponding funcitonal is

$$F = x\sqrt{1+\dot{y}^2} + M\sqrt{1+\dot{y}^2}$$

which results in the solution

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right) - M$$

$$\left(a \cdot \sinh\left(\frac{x-b}{a}\right)\right)_0^{10} = 11$$

Set up a cost function

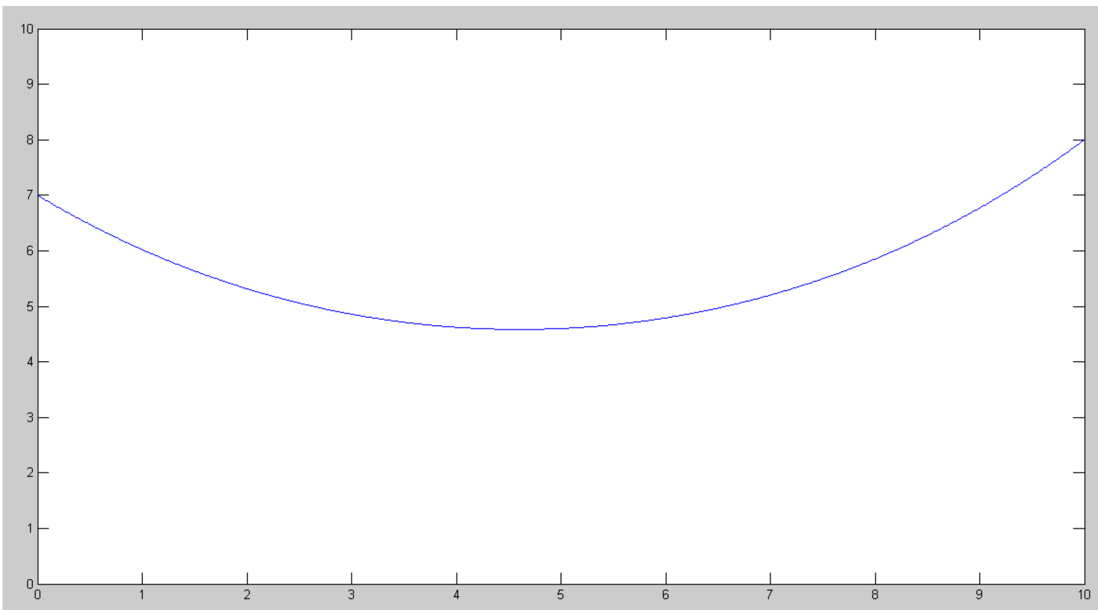
```
function J = chain(z)
a = z(1);
b = z(2);
M = z(3);

Length = 11;
x1 = 0;
y1 = 7;

x2 = 10;
y2 = 8;

e1 = a*cosh((x1-b)/a) - M - y1;
e2 = a*cosh((x2-b)/a) - M - y2;
e3 = a*sinh((x2-b)/a) - a*sinh((x1-b)/a) - Length;

J = e1^2 + e2^2 + e3^2;

end
```

Solving

```
>> [Z,e] = fminsearch('chain',[1,2,3])

          a        b          M
Z =    6.6998    4.3893    1.1897

e =  4.2875e-010

>>
```

plotting the shape

```
>> a = z(1);
>> b = z(2);
>> M = z(3);
>> x = [0:0.01:10]';
>> y = a*cosh( (x-b)/a ) - M;
>> plot(x,y);
>> ylim([0,10])
```



Ricatti Equation

4) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^{10} (4x^2 + 2\dot{x}^2) dt$$

x(0) = 5

x(10) = 6

The solution must satisfy the Euler LaGrange equation

$$F_x - \frac{d}{dt}(F_{x'}) = 0$$

$$8x - \frac{d}{dt}(4\dot{x}) = 0$$

Doing some algebra

$$8x - 4\ddot{x} = 0$$

$$(2 - s^2)x = 0$$

$$s = \pm\sqrt{2}$$

so the answwer is of the form

$$x(t) = ae^{1.414t} + be^{-1.414t}$$

Plug in the endpoint constraints

$$x(0) = 5 = a + b$$

$$x(10) = 6 = ae^{14.14} + be^{-14.14}$$

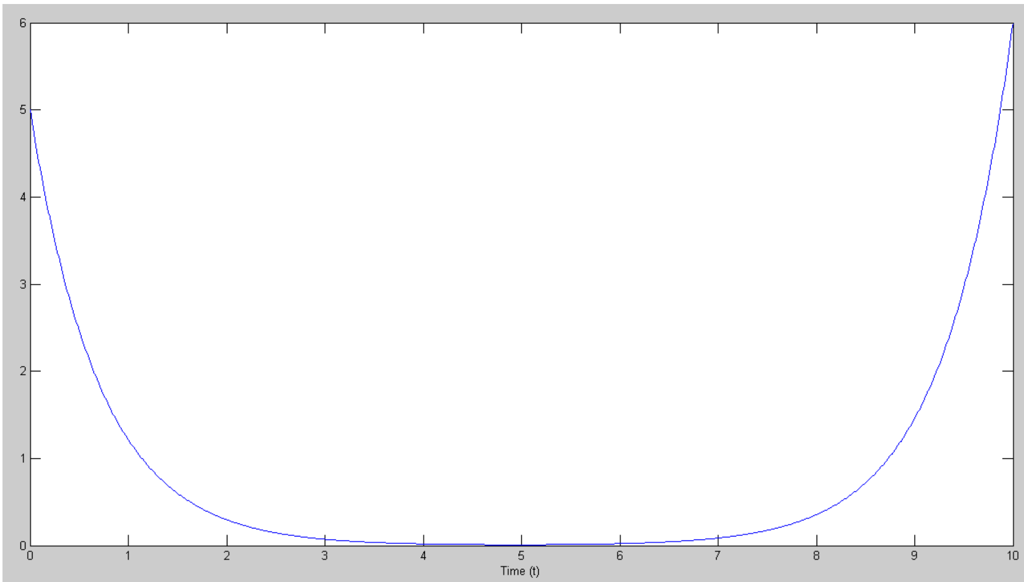
Solving

```
>> p1 = sqrt(2);
>> p2 = -p1;
>> B = [1,1 ; exp(10*p1),exp(10*p2)];
>> A = [1,1 ; exp(10*p1),exp(10*p2)];
>> B = [5;6];
>> inv(A)*B

a   0.000004328122314
b   4.999995671877686

>>
```

$$x(t) = 0.000004328e^{\sqrt{2}\,t} + 4.99999567e^{-\sqrt{2}\,t}$$

Plotting x(t) in Matlab:

```
>> p1 = sqrt(2);
>> p2 = -p1;
>> B = [1,1 ; exp(10*p1),exp(10*p2)];
>> A = [1,1 ; exp(10*p1),exp(10*p2)];
>> B = [5;6];
>> C = inv(A)*B;
>> a = C(1);
>> b = C(2);
>> t = [0:0.01:10]';
>> x = a*exp(p1*t) + b*exp(p2*t);
>> plot(t,x)
>> xlabel('Time (t)')
```



5) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^{10} (4x^2 + 2u^2) dt$$

$$\dot{x} = -0.1x + u$$

x(0) = 5

x(10) = 6

The functional for this problem (including a LaGrange multiplier) is

$$F = (4x^2 + 2u^2) + m(\dot{x} + 0.1x - u)$$

Solving three Euler LaGrange equations

$$F_x - \tfrac{d}{dt}(F_{x'}) = 0$$

$$(8x + 0.1m) - \tfrac{d}{dt}(m) = 0$$

(1) $$8x + 0.1m - \dot{m} = 0$$

$$F_u - \tfrac{d}{dt}(F_{u'}) = 0$$

(2) $$4u - m = 0$$

$$F_m - \tfrac{d}{dt}(F_{m'}) = 0$$

(3) $$\dot{x} = -0.1x + u$$

Substituting

$$m = 4u \qquad (2)$$

$$8x + 0.1(4u) - (4\dot{u}) = 0 \qquad \text{plug (2) into (1)}$$

$$u = \dot{x} + 0.1x \qquad (3)$$

$$8x + 0.4(\dot{x} + 0.1x) - 4\tfrac{d}{dt}(\dot{x} + 0.1x) = 0$$

$$8x + 0.4(\dot{x} + 0.1x) - 4(\ddot{x} + 0.1\dot{x}) = 0$$

$$-4\ddot{x} + 8.04x = 0$$

$$(-4s^2 + 8.04)x = 0$$

$$s = \pm 1.4177$$

So, x(t) is of the form

$$x(t) = ae^{1.4177t} + be^{-1.4177t}$$

Plug in the endpoinnt constraints

$$x(0) = 5 = a + b$$

$$x(10) = 6 = ae^{14.177} + be^{-14.177}$$

Soving (code below):

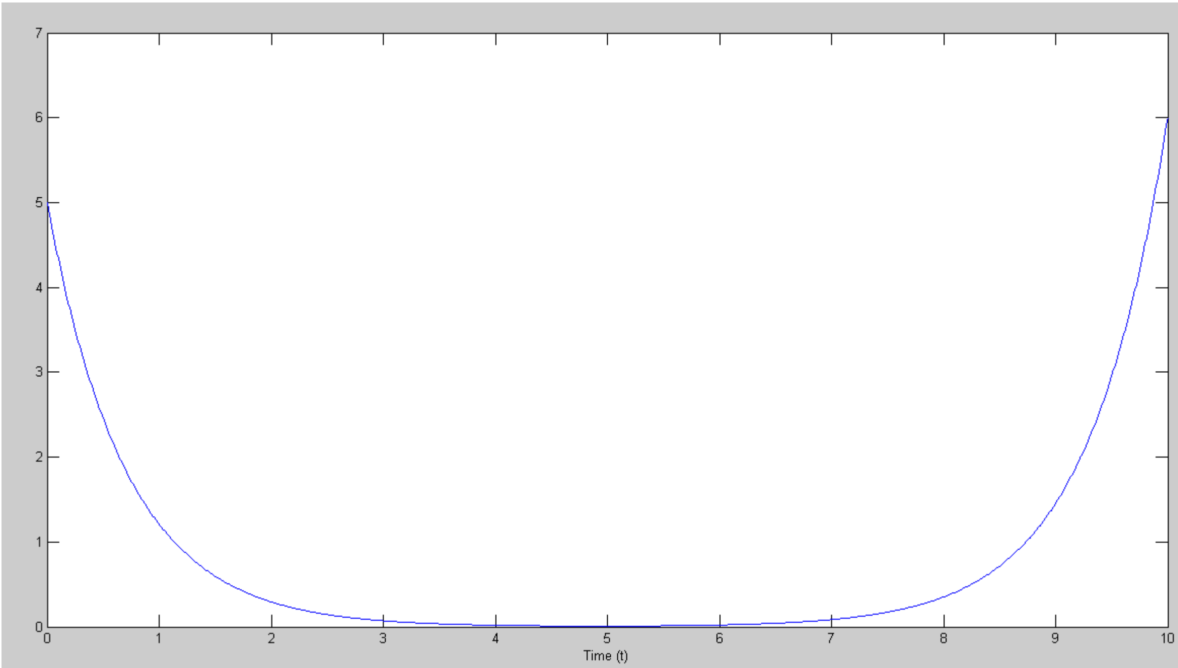
$$x(t) = 0.000004177e^{1.4177t} + 4.999995822e^{-1.4177t}$$

The optimal input comes from

$$u = \dot{x} + 0.1x$$

Plotting this in Matlab

```
>> p = (8.04/4)^0.5

p =

    1.4177

>> A = [1,1 ; exp(10*p),exp(-10*p)];
>> B = [5;6];
>> C = inv(A)*B

C =

    0.0000
    5.0000

>> format long
>> C

C =

   0.000004177957823
   4.999995822042177

>> t = [0:0.01:10]';
>> a = C(1);
>> b = C(2);
>> y = a*exp(p*t) + b*exp(-p*t);
>> plot(t,y)
>> xlabel('Time (t)')
>>
```



Optimal Path for x(t)