LaGrangian Formulation of System Dynamics

Find the dynamics of a nonlinear system:

Circuit analysis tools work for simple lumped systems. For more complex systems, especially nonlinear ones, this approach fails. The Lagrangian formulation for system dynamics is a way to deal with any system.

Definitions:

KE Kinetic Energy in the system

PE Potential Energy

$\frac{\partial}{\partial t}$ The partial derivative with respect to 't'. All other variables are treated as constants.

$\frac{d}{dt}$ The full derivative with respect to t.

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \ldots$$

L Lagrangian = KE - PE

Procedure:

1) Define the kinetic and potential energy in the system.

2) Form the Lagrangian:

$$L = KE - PE$$

3) The input is then

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

where F_i is the input to state x_i. Note that

- If x_i is a position, F_i is a force.
- If x_i is an angle, F_i is a torque

Example:

Example: Determine the dynamics of a rocket

Step 1: Determine the potential and kinetic energy of the rocket

Potential Energy

$$PE = mgx$$

Kinetic Energy:

$$KE = \frac{1}{2}m\dot{x}^2$$

Step 2: Set up the LaGrangian

$$L = KE - PE$$

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

Step 3: Take the partials

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

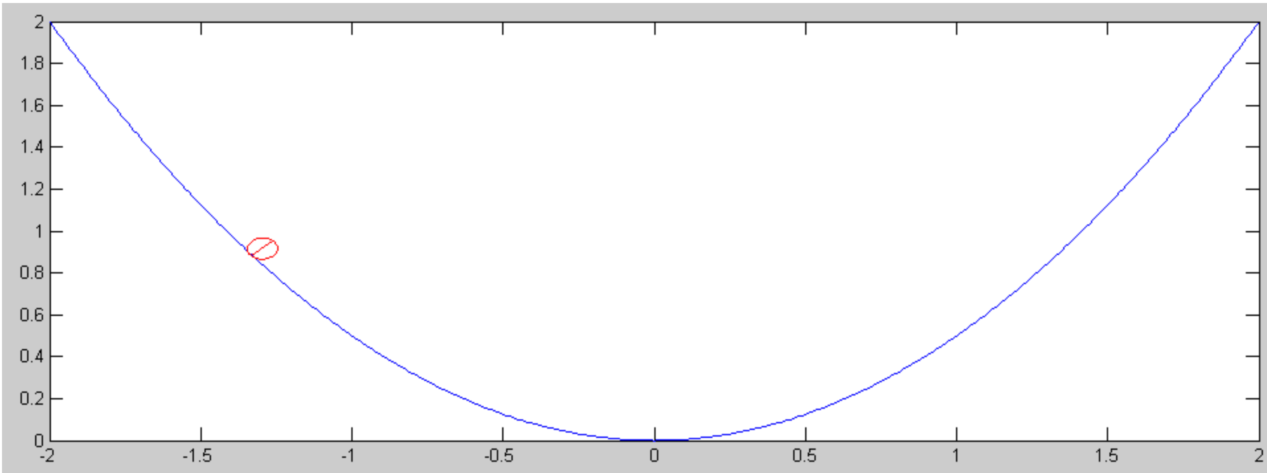
$$F = \frac{d}{dt}(m\dot{x}) - (-mg)$$

Take the full derivative with respect to t

$$\boxed{F = m\ddot{x} + \dot{m}\dot{x} + mg}$$

Note that if the rocket is loosing mass you get the term $\dot{m}\dot{x}$. If you leave this term out, the rocket misses the target.

Example 2: Ball in a parabolic bowl



Determine the dynamics of a ball rolling in a bowl characterized by

$$y = \frac{1}{2}x^2$$

Step 1: Define the kinetic and potential energy

Potential Energy:

$$PE = mgy = \frac{1}{2}mgx^2$$

Kinetic Energy: This has two terms, one for translation and one for rotation .

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}J\dot{\theta}^2$$

The velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

The rotational velocity is

$$position = r\theta$$

$$v = r\dot{\theta}$$

Note that

$$y = \frac{1}{2}x^2$$

$$\dot{y} = x\dot{x}$$

gives

$$KE = \tfrac{1}{2}mv^2 + \tfrac{1}{2}J\left(\tfrac{v}{r}\right)^2$$

$$KE = \tfrac{1}{2}\left(m + \tfrac{J}{r^2}\right)v^2$$

$$KE = \tfrac{1}{2}\left(m + \tfrac{J}{r^2}\right)(\dot{x}^2 + \dot{y}^2)$$

$$KE = \tfrac{1}{2}\left(m + \tfrac{J}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$

The inertia depends upon what type of ball you are using:

$J = 0$	point mass with all the mass in the center
$J = \frac{2}{5}mr^2$	solid sphere
$J = \frac{2}{3}mr^2$	hollow sphere
$J = mr^2$	hollow cyllinder

Assume the ball is a solid sphere

$$KE = \tfrac{1}{2}\left(m + \frac{\frac{2}{5}mr^2}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$

$$KE = 0.7m(1^2 + x^2)\dot{x}^2$$

Step 2: Form the LaGrangian

$$L = KE - PE$$

$$L = 0.7m(1^2 + x^2)\dot{x}^2 - \tfrac{1}{2}mgx^2$$

Step 3: Take the partials. The partial with respect to x is:

$$\frac{\partial L}{\partial x} = 0.7m(2x)\dot{x}^2 - mgx$$

$$\frac{\partial L}{\partial x} = 1.4mx\dot{x}^2 - mgx$$

The partial with respect to dx/dt is:

$$\frac{\partial L}{\partial \dot{x}} = 1.4m(1^2 + x^2)\dot{x}$$

The full derivative of the partial with respect to dx/dt is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(1.4m(1^2 + x^2)\dot{x})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 1.4m(2x\dot{x})\dot{x} + 1.4m(1^2 + x^2)\ddot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 2.8mx\dot{x}^2 + 1.4m(1^2 + x^2)\ddot{x}$$

So, the dynamics are:

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = (2.8mx\dot{x}^2 + 1.4m(1^2 + x^2)\ddot{x}) - (1.4mx\dot{x}^2 - mgx)$$

$$F = 1.4mx\dot{x}^2 + 1.4m(1^2 + x^2)\ddot{x} + mgx$$

In free fall, F = 0. Solving for the highest derrivative:

$$\boxed{\ddot{x} = -\left(\frac{\left(1.4\dot{x}^2 + g\right)x}{1.4\left(1^2 + x^2\right)}\right)}$$

Matlab Code (Ball.m)

```
% Dynamics of a ball rolling in a bowl where
%
%    y = 0.5 x^2
%

x = 1.5;
dx = 0;
dt = 0.01;
t = 0;
while(t < 100)

% compute the acceleration

ddx = -( 1.4*dx*dx + 9.8) * x / ( 1.4*(1 + x*x) );

% integrate

x = x + dx*dt;
dx = dx + ddx*dt;

% display the ball
 y = 0.5*x*x;

 x1 = [-2:0.01:2]';
 y1 = 0.5* (x1 .^ 2);

 % draw the ball
 i = [0:0.01:1]' * 2 * pi;
 xb = 0.05*cos(i) + x;
 yb = 0.05*sin(i) + 0.5*x^2 + 0.05 + 0.02*abs(x);

 % line through the ball
 q = [0, pi] - x/0.05;
 xb1 = 0.05*cos(q) + x;
 yb1 = 0.05*sin(q) + 0.5*x^2 + 0.05 + 0.02*abs(x);

 plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');

 pause(0.01);
 end
```