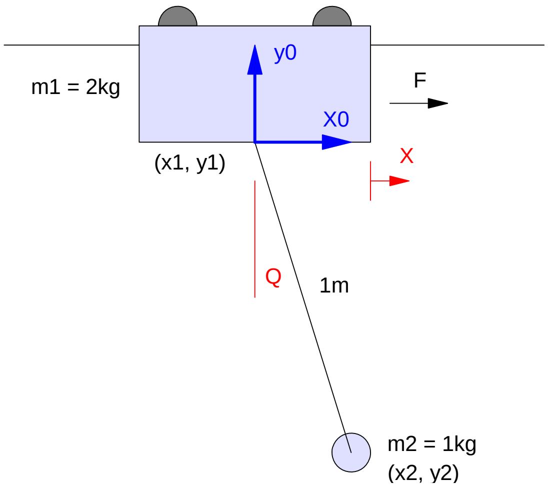
# Gantry & Cart and Pendulum Dynamics

Find the dynamics for the following system: A 1kg mass swings from a gantry which weighs 2kg. The length of the rope is 1m.




Gantry System: Zeroth Refernce Frame = $(x_0, y_0)$

1) Write the energy in the system

Mass #1 (cart): position = $(x_1, y_1)$

- $x_1 = x$
- $y_1 = 0$

$$KE_1 = \frac{1}{2}mv^2 = \frac{1}{2}\cdot 2\cdot \dot{x}^2$$

$$PE_1 = mgh = 0$$

Mass #2 (pendulum): position = $(x_2, y_2)$

- $x_2 = x_1 + \sin\theta$ $\quad\quad \dot{x}_2 = \dot{x}_1 + (\cos\theta)\dot{\theta}$
- $y_2 = -\cos\theta$ $\quad\quad \dot{y}_2 = (\sin\theta)\dot{\theta}$

$$KE_2 = \frac{1}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$KE_2 = \frac{1}{2}\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}_1\cos\theta + \cos^2\theta\cdot\dot{\theta}^2 + \sin^2\theta\cdot\dot{\theta}^2\right)$$

$$KE_2 = \frac{1}{2}\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}_1\cos\theta + \dot{\theta}^2\right)$$

$$PE_2 = mgh = gy_2$$

$$PE_2 = -g\cos\theta$$

So, the Lagrangian is

$$L = \left(\dot{x}^2 + \tfrac{1}{2}\left(\dot{x}^2 + 2\dot{x}\dot{\theta}\cos\theta + \dot{\theta}^2\right)\right) - (-g\cos\theta)$$

$$\boxed{L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta}$$

The force on the base is

$$L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(3\dot{x} + \dot{\theta}\cos\theta\right) - 0$$

$$\boxed{F = 3\ddot{x} + \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta}$$

The torque on the rod is

$$L = \left(\tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2}\dot{\theta}^2\right) + g\cos\theta$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(\dot{x}\cos\theta + \dot{\theta}\right) - \left(-\dot{x}\dot{\theta}\sin\theta - g\sin\theta\right)$$

$$T = \ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta + \ddot{\theta} + \dot{x}\dot{\theta}\sin\theta + g\sin\theta$$

$$\boxed{T = \ddot{x}\cos\theta + \ddot{\theta} + g\sin\theta}$$

So, the dynamics are

$$\boxed{\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2\sin\theta \\ -g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}F}$$

## Linear Model

For small perturbations about 0,

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1$$

$$\dot{\theta}^2 \approx 0$$

g = 9.8 m/s$^2$

The dynamics linearized about zero are then

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 0 \\ 9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F\right)$$

or

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 4.9\theta + 0.5F \\ -14.7\theta - 0.5F \end{bmatrix}$$

which is a linear differential equation. You can put this in state-space form:

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4.9 & 0 & 0 \\ 0 & -14.7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

Note that the eigenvalues of 'A' are

$$\{0, 0, j3.83, -j3.83\}$$

corresponding to an oscillatory system as expected. Given an initial condition, the pendulum swings back and forth at 3.83 rad/sec.

The eigenvectors are:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.316 \\ -0.950 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.316 \\ -0.950 \end{pmatrix} \right\} \qquad \text{note: states are } \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$

This means

- If you displace the cart by +1m, that state decays as $e^{0t}$ ( first eigenvector )
- If you start the cart moving right at 1m/s, it keeps drifting right decaying as $e^{0t}$ ( 2nd eigenvector )
- If you displace the cart 0.316m right and swing the angle 0.950 rad left, the cart oscillates back and forth at 3.83 rad/sec ( 3rd eigenvector )
- If yo start out at (0, 0) but make the initia velocity 0.316 m/s right and the initial angular velocity -0.950 rad/sec, the cart osicllates back and forth at 3.83 rad/sec ( 4th eigenvector )

If you invert gravity:

g = -9.8 m/s$^2$:

then you get an inverted pendulum:

$$s \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9 & 0 & 0 \\ 0 & 14.7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

results in the poles being

$$\{0, 0, 3.83, -3.83\}$$

As expected, an inverted pendulum is unstable.