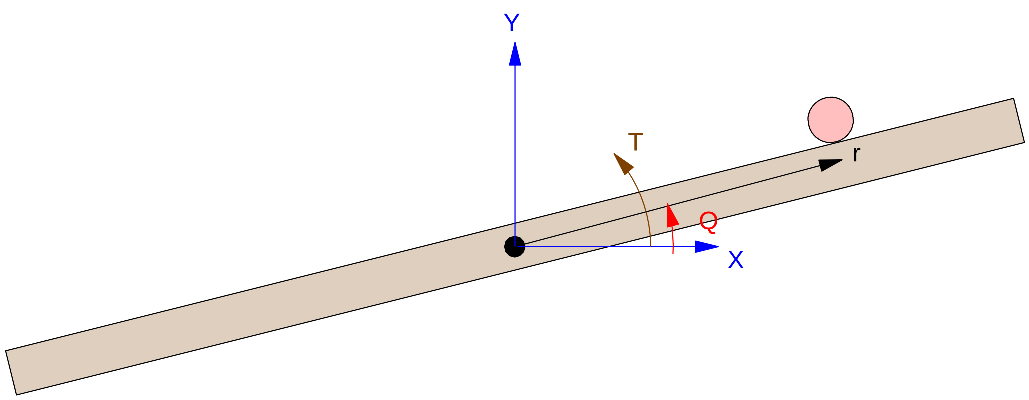
Ball and Beam System

Find the dynamics for the following system: A 1kg ball rolls along a beam of length L. The nominal position of the ball is at 0.5 meters.

- The ball has a mass of 1kg
- The beam has a rotational inertia of 0.2 kg m2
- A motor applies a torque to the beam (T).
- The goal is to balance the ball at a certain spot (1.0 meters in this case).



1) Write the position and velocity of the ball

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

2) Write the energy in the system

$$PE = mgy = mgr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{5}m\dot{r}^2$$

Note that the kinetic energy has three terms:

- The rotational energy of the beam
- The translational energy of the ball, and
- The rotational energy of the ball. Assume a solid sphere.

Substituting...

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{5}m\dot{r}^2$$

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + \frac{1}{5}m\dot{r}^2$$

$$KE = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{5}m\dot{r}^2$$

Pluggin in numbers (J = 0.2, m = 1)

$$KE = 0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2$$

So, the LaGrangian is

$$L = KE - PE$$

$$\boxed{L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)}$$

The force on the ball (i.e. if there was a motor driving the ball or friction)

$$L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(1.4\dot{r}) - \left(r\dot{\theta}^2 - g\sin\theta\right)$$

$$\boxed{F = 1.4\ddot{r} - r\dot{\theta}^2 + g\sin\theta}$$

The torque on the rod is

$$L = \left(0.1\dot{\theta}^2 + 0.7\dot{r}^2 + 0.5r^2\dot{\theta}^2\right) - (gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(0.2\dot{\theta} + r^2\dot{\theta}\right) - (-gr\cos\theta)$$

$$\boxed{T = 0.2\ddot{\theta} + r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta}$$

Putting it together:

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.2+r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearized Model: At r = 1.0, angle = zero, F = 0, m = 1kg

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 1.2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -g\theta \\ -gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

In state-space

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -8.167 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.833 \end{bmatrix} T$$

The eigenvalues and eigenvectors are:

```
>> A = [0,0,1,0 ; 0,0,0,1 ; 0,-7,0,0 ; -8.167,0,0,0];
>> eig(A)

  -2.7497
  -0.0000 + 2.7497i
  -0.0000 - 2.7497i
   2.7497
```

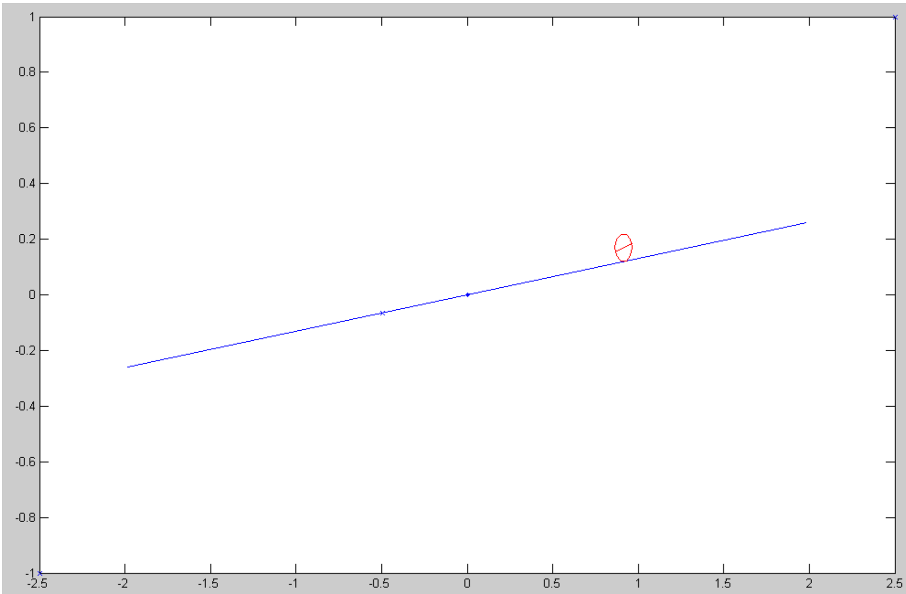
The eigenvectors are:

```
>> [a,b] = eig(A)

  -0.2322              0.0000 + 0.2322i   0.0000 - 0.2322i  -0.2322
   0.2508             -0.0000 + 0.2508i  -0.0000 - 0.2508i   0.2508
   0.6384             -0.6384 + 0.0000i  -0.6384 - 0.0000i  -0.6384
  -0.6896             -0.6896            -0.6896             0.6896
```

This is a difficult system to control since you an unstable pole and two complex poles. Trying to control the torque by hand to get the ball to a certain position is very hard.

Matlab Animation Files:



Main Callint Routine (Beam.m)

```
X = [0.1, 0, 0, 0]';
Ref = 0.2;
dt = 0.01;
t = 0;

% magic we'll get to later
Kx = [-4.3058, 28.8754,-10.6567, 5.1004];
Kr = -13.7652;

while(t < 20)

 Ref = sign(sin(t)) * 0.5;

 U = Kr*Ref - Kx*X;

 dX = BeamDynamics(X, U);

 X = X + dX * dt;
 t = t + dt;

 if (X(1) > 2)
     X(3) = -abs(X(3));
     end
 if (X(1) < -2)
     X(3) = abs(X(3));
 end

 BeamDisplay(X, Ref);

end
```

Beam Dynamics Routine

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.2+r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 - g\sin\theta \\ -2r\dot{r}\dot{\theta} - gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

```
function [dX] = BeamDynamics( X, T )

r = X(1);
q = X(2);
dr = X(3);
dq = X(4);

g = 9.8;
F = 0;

M = [1.4, 0; 0, 0.2+r*r];
B1 = F + r*dq*dq - g*sin(q);
B2 = T - 2*r*dr*dq - g*r*cos(q);

dZ = inv(M)*[B1; B2];
dX = [dr; dq; dZ];

end
```

Beam Display Routine

```
function [] = BeamDisplay( X, Ref )
x = X(1);
q = X(2);
dx = X(3);
dq = X(4);

% draw the beam

x1 = -2*cos(q);
y1 = -2*sin(q);
x2 = 2*cos(q);
y2 = 2*sin(q);

clf;
plot([-2.5,2.5],[-1,1],'x');
hold on

plot([x1,x2],[y1,y2],'b');
plot(0,0,'.');

% draw the ball

i = [0:0.1:1]' * 2*pi;
xb = x*cos(q) + 0.05*cos(i)  - 0.05*sin(q);
yb = x*sin(q) + 0.05*sin(i)  + 0.05*cos(q);
plot(xb,yb,'r');

% draw a line through the ball so you can see it roll

Q = -x / 0.05;
x1 = x*cos(q) - 0.05*sin(q) + 0.05*cos(Q);
y1 = x*sin(q) + 0.05*cos(q) + 0.05*sin(Q);

Q = Q + pi;
x2 = x*cos(q) - 0.05*sin(q) + 0.05*cos(Q);
y2 = x*sin(q) + 0.05*cos(q) + 0.05*sin(Q);
plot([x1,x2],[y1,y2],'r-');

plot(Ref*cos(q),Ref*sin(q),'bx');



pause(0.01);
end
```