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# **Dynamics of a 2-Link Arm**

**NDSU ECE 463/663**

**Lecture #9**

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Please visit [Bison Academy](#) for corresponding  
lecture notes, homework sets, and solutions



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# Robotic Arms

<https://blog.technavio.com/wp-content/uploads/2018/07>

Industrial robots typically have 3 degrees of freedom

- base - shoulder - elbow
- Allows you to reach any spot in 3-space

Plus 3 degrees of freedom at the tip

- Wrist
- Allows any orientation at the tip

LaGrange formulation of dynamics works

- About a 10-page derivation



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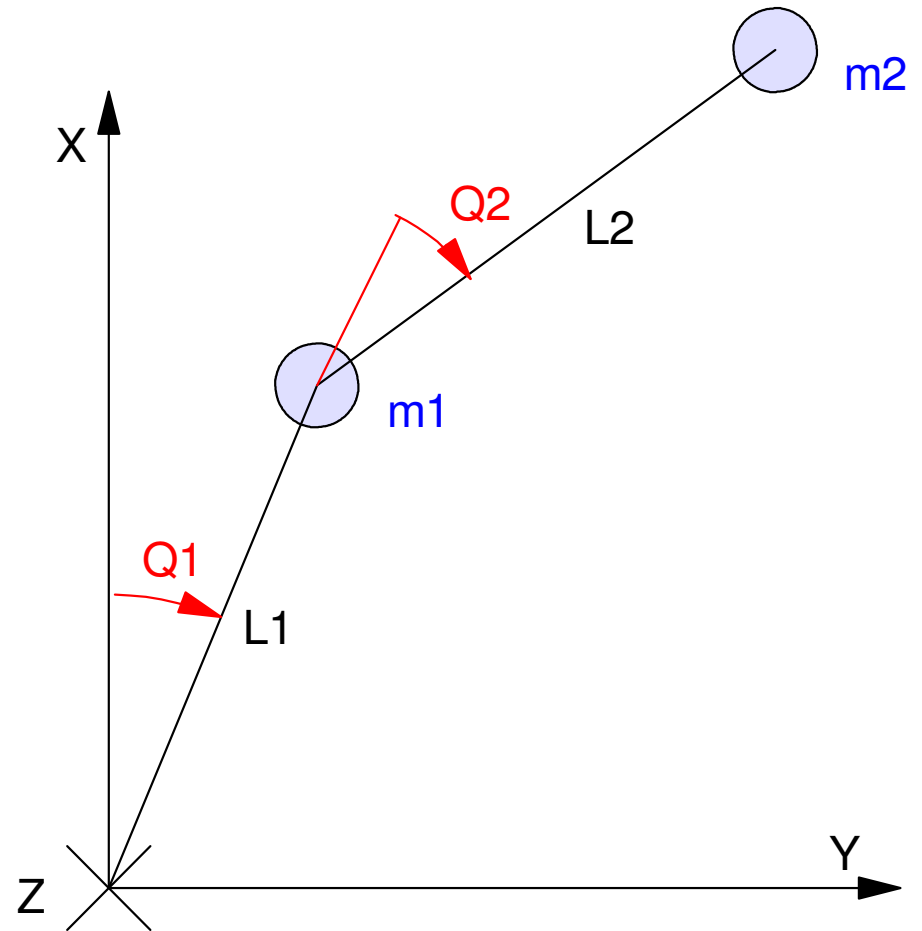
## 2-Link Arm

- Ignore the wrist
- Fix the base at zero degrees

Results in a 2-link robotic arm

- The robot can move about the XY plane
- Zero position points straight out

Tip =  $f(Q1, Q2)$



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## 2-Link Dynamics:

Determine the dynamics

- $L1 = L2 = 1\text{m}$
- $m1 = 1\text{kg}$
- $m2 = 1\text{kg}$

Step 1: Determine the (x,y) position and velocity of each mass

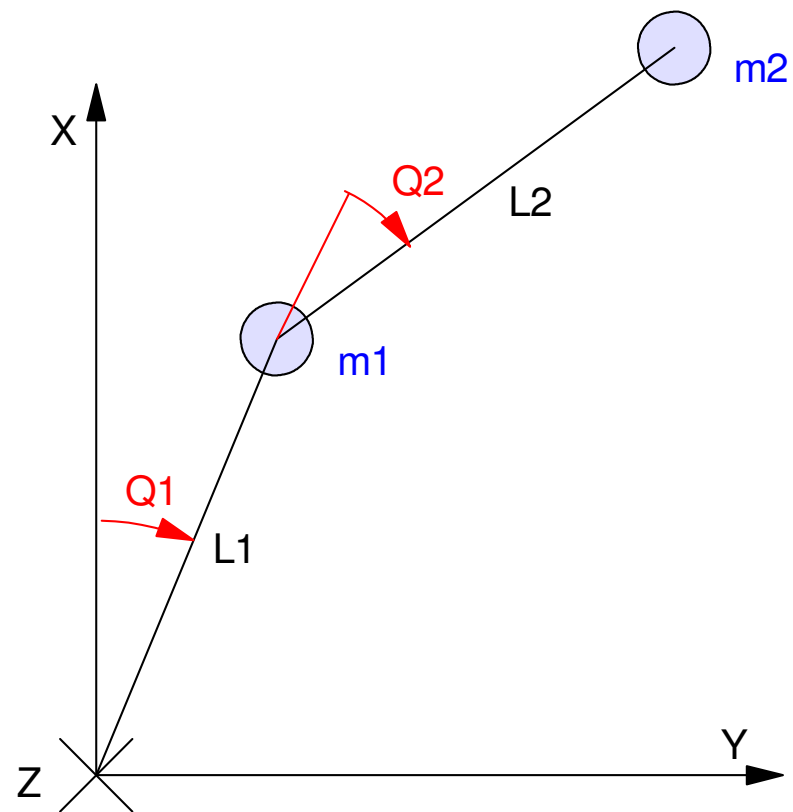
mass 1:

$$x_1 = \cos \theta_1$$

$$\dot{x}_1 = -\sin(\theta_1)\dot{\theta}_1$$

$$y_1 = \sin(\theta_1)$$

$$\dot{y}_1 = \cos(\theta_1)\dot{\theta}_1$$



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mass 2:

$$x_2 = x_1 + \cos(\theta_1 + \theta_2)$$

$$\dot{x}_2 = -\sin(\theta_1)\dot{\theta}_1 - \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

$$y_2 = y_1 + \sin(\theta_1 + \theta_2)$$

$$\dot{y}_2 = \cos(\theta_1)\dot{\theta}_1 + \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

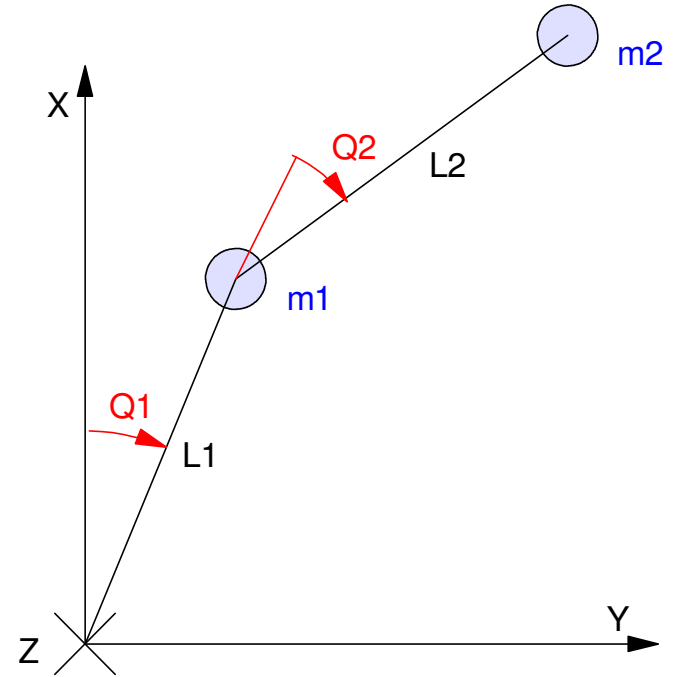
Using shorthand notation

$$x_1 = c_1 \quad \dot{x}_1 = -s_1\dot{\theta}_1$$

$$y_1 = s_1 \quad \dot{y}_1 = c_1\dot{\theta}_1$$

$$x_2 = c_1 + c_{12} \quad \dot{x}_2 = -s_1\dot{\theta}_1 - s_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

$$y_2 = s_1 + s_{12} \quad \dot{y}_2 = c_1\dot{\theta}_1 + c_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$



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Step 2: Form the kinetic and potential energy:

Mass 1:  $m_1 = 1$

$$KE = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2)$$

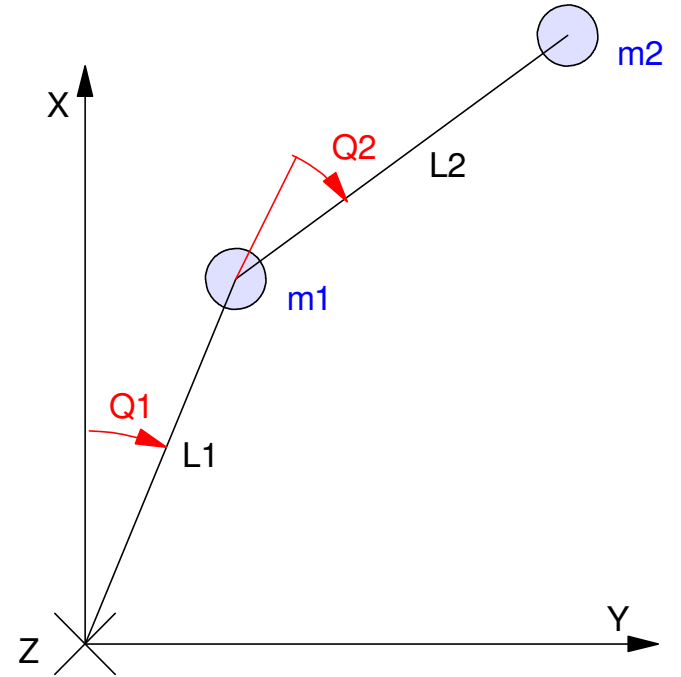
$$KE = \frac{1}{2}\left(\left(-s_1\dot{\theta}_1\right)^2 + \left(c_1\dot{\theta}_1\right)^2\right)$$

Note that  $\sin^2 + \cos^2 = 1$

$$KE = \frac{1}{2} \dot{\theta}_1^2$$

$$PE = m_1 g x_1$$

$$PE = g c_1$$



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Mass 2:  $m_2 = 1$

$$KE = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$KE = \frac{1}{2}\left(\left(-s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2 + \left(c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2\right)$$

$$KE = \frac{1}{2}\left(s_1^2\dot{\theta}_1^2 + s_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + c_1^2\dot{\theta}_1^2 + c_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right)$$

$$+ s_1s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_1c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = \frac{1}{2}\left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) + c_1c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_1s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

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From trigonometry

$$\sin(a)\sin(b) + \cos(a)\cos(b) = \cos(a - b)$$

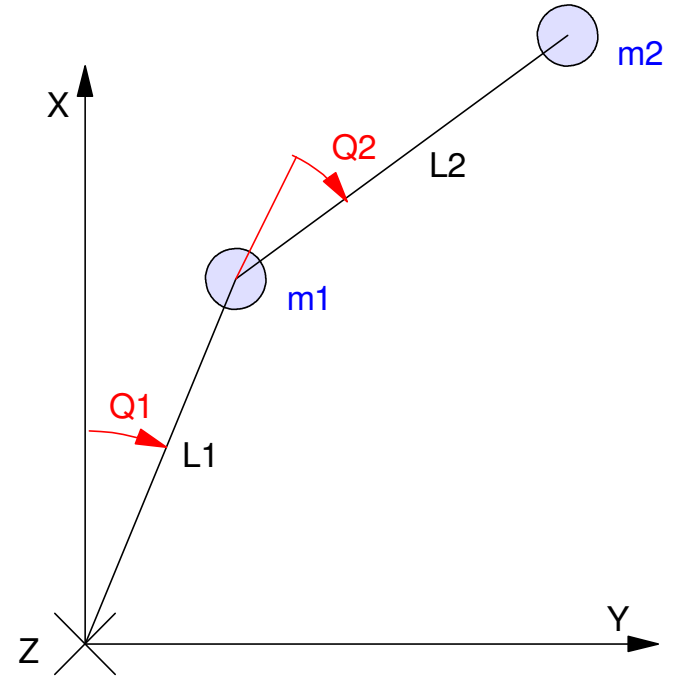
resulting in

$$KE = \frac{1}{2} \left( \dot{\theta}_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 \right) + c_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$KE = (1 + c_2) \dot{\theta}_1^2 + \frac{1}{2} \dot{\theta}_2^2 + (1 + c_2) \dot{\theta}_1 \dot{\theta}_2$$

$$PE = m_2 g x_2$$

$$PE = g(c_1 + c_{12})$$





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Step 3: Form the LaGrangian:

$$L = KE - PE$$

$$L = \left( \left( \frac{1}{2} \dot{\theta}_1^2 \right) + \left( (1 + c_2) \dot{\theta}_1^2 + \frac{1}{2} \dot{\theta}_2^2 + (1 + c_2) \dot{\theta}_1 \dot{\theta}_2 \right) \right) - ((gc_1) + (g(c_1 + c_{12})))$$

$$L = (1.5 + c_2) \dot{\theta}_1^2 + \frac{1}{2} \dot{\theta}_2^2 + (1 + c_2) \dot{\theta}_1 \dot{\theta}_2 - 2gc_1 - gc_{12}$$

Step 4: Take the partials

$$T_1 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \left( \frac{\partial L}{\partial \theta_1} \right)$$

$$T_2 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \left( \frac{\partial L}{\partial \theta_2} \right)$$

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Starting with  $\theta_1$ :

$$L = (1.5 + c_2)\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_1 = \frac{d}{dt}\left((3 + 2c_2)\dot{\theta}_1 + (1 + c_2)\dot{\theta}_2\right) - (-2gs_1 - gs_{12})$$

$$T_1 = \left((3 + 2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1 + c_2)\ddot{\theta}_2 - s\right)$$

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Moving on to  $\theta_2$

$$L = (1.5 + c_2)\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 - 2gc_1 - gc_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

$$T_2 = \frac{d}{dt}\left(\dot{\theta}_2 + (1 + c_2)\dot{\theta}_1\right) - \left(-s_2\dot{\theta}_1^2 - s_2\dot{\theta}_1\dot{\theta}_2 - gs_{12}\right)$$

$$T_2 = \left(\ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) + \left(s_2\dot{\theta}_1^2 + s_2\dot{\theta}_1\dot{\theta}_2 + gs_{12}\right)$$

$$T_2 = \ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gs_{12}$$

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Net result:

$$T_1 = \left( (3 + 2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1 + c_2)\ddot{\theta}_2 - s_2\dot{\theta}_2^2 \right) + (2gs_1 + gs_{12})$$

$$T_2 = \ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 + gs_{12}$$

To solve, rewrite this in matrix form:

$$\begin{bmatrix} (3 + 2c_2) & (1 + c_2) \\ (1 + c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix}$$

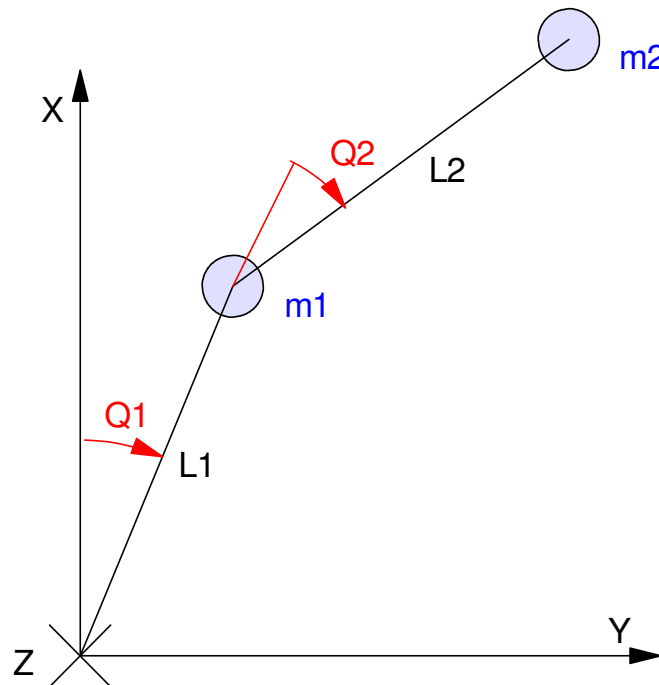
Mass Matrix \* Acceleration = Torque + Coriolis Forces + Gravity

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Given the joint angles, velocities, gravity, and input torques, you can compute the joint accelerations as

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (3 + 2c_2) & (1 + c_2) \\ (1 + c_2) & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$



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## MatLab Code:

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 1s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$

```
function [ ddQ ] = TwoLinkDynamics( Q, dQ, T )
```

```
q1 = Q(1);  
q2 = Q(2);  
dq1 = dQ(1);  
dq2 = dQ(2);  
c1 = cos(q1);  
s1 = sin(q1);  
s2 = sin(q2);  
c2 = cos(q2);  
s12 = sin(q1+q2);  
c12 = cos(q1+q2);  
g = 9.8;  
M = [ 3 + 2*c2 , 1+c2 ; 1+c2 , 1];  
C = [ 2*s2*dq1*dq2 + s2*dq2*dq2 ; -s2*dq1*dq1 ];  
G = g*[2*s1 + s12 ; s12 ];  
ddQ = inv(M) * ( T + C + G );  
end
```

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## 2-Link Arm in Freefall

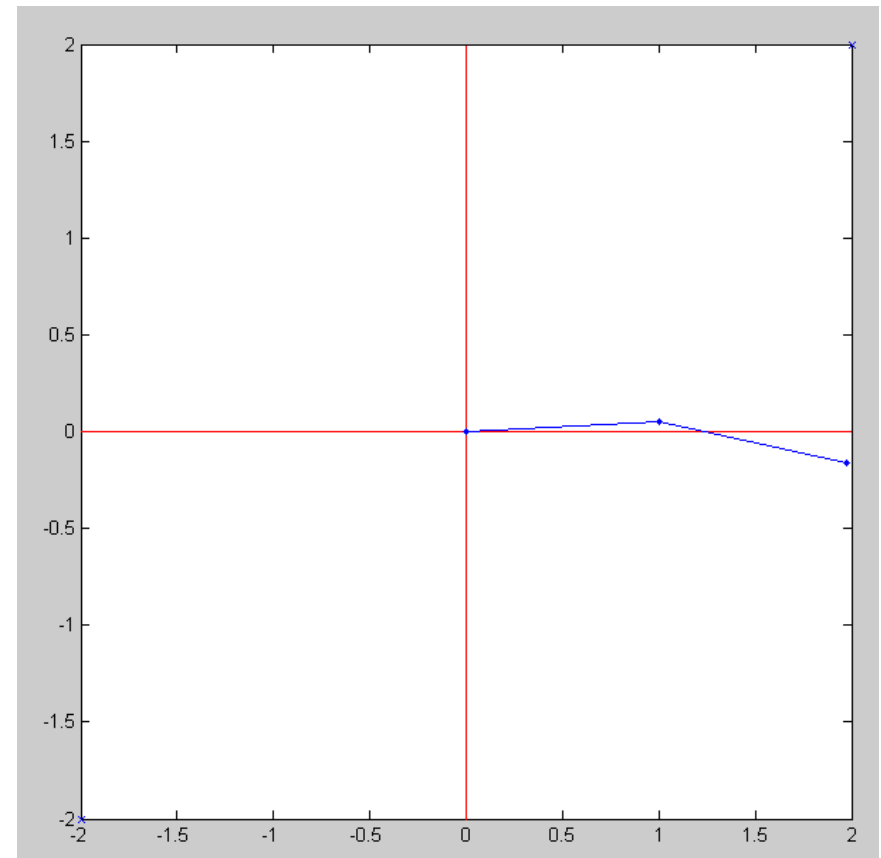
```
% 2-Link robot in free-fall (T=0)
% rev 6/5/20

Q = [1; 0];
dQ = [0; 0];
T = [0; 0];
t = 0;
dt = 0.01;
data = [];
while(t < 10)

    c1 = cos(Q(1));
    s1 = sin(Q(1));
    c2 = cos(Q(2));
    s2 = sin(Q(2));

% Freefall
    T = [0;0];

    ddQ = TwoLinkDynamics(Q, dQ, T);
    dQ = dQ + ddQ * dt;
    Q = Q + dQ*dt;
    t = t + dt;
```



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```
% Plot the Robot
```

```
    x0 = 0;  
    y0 = 0;  
    x1 = cos(Q(1));  
    y1 = sin(Q(1));  
    x2 = x1 + cos(Q(1) + Q(2));  
    y2 = y1 + sin(Q(1) + Q(2));  
    clf;  
    plot([-2,2],[-2,2],'x');  
    hold on;  
    plot([-2,2],[0,0], 'r');  
    plot([0,0],[-2,2], 'r');  
    plot([y0, y1, y2], [x0, x1, x2], 'b.-');  
    pause(0.01);  
    data = [data ; x2, y2];  
end
```

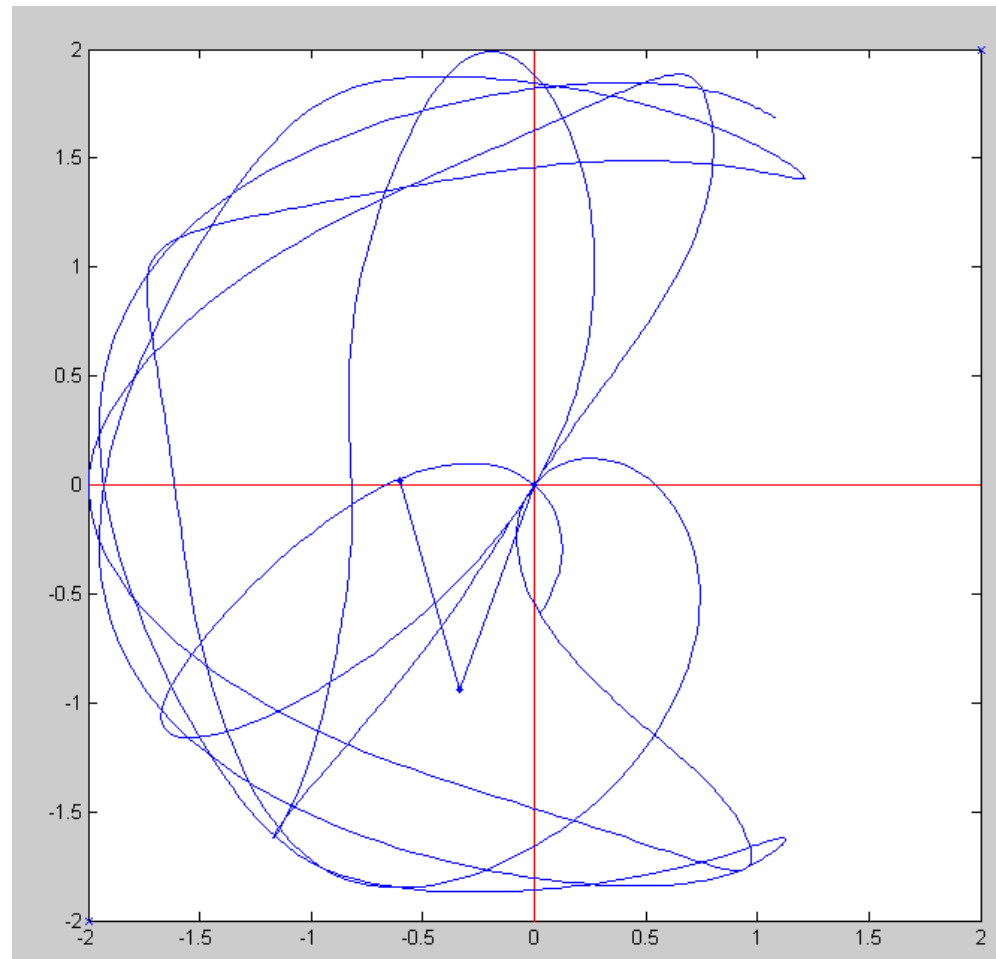
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Note: This is a chaotic system

- Small changes in the initial conditions results in large changes in the trajectory



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# Butterfly Effect

[https://en.wikipedia.org/wiki/Butterfly\\_effect](https://en.wikipedia.org/wiki/Butterfly_effect)

*In chaos theory, the butterfly effect is the sensitive dependence on initial conditions in which a small change in one state of a deterministic nonlinear system can result in large differences in a later state.*

*The term butterfly effect is closely associated with the work of Edward Lorenz. It is derived from the metaphorical example of the details of a tornado (the exact time of formation, the exact path taken) being influenced by minor perturbations such as a distant butterfly flapping its wings several weeks earlier. Lorenz discovered the effect when he observed that runs of his weather model with initial condition data that were rounded in a seemingly inconsequential manner. He noted that the weather model would fail to reproduce the results of runs with the unrounded initial condition data. A very small change in initial conditions had created a significantly different outcome.[1]*

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## Linearized Dynamics about (0,0)

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (3+2c_2) & (1+c_2) \\ (1+c_2) & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 1s_1 + s_{12} \\ s_{12} \end{bmatrix} \right)$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = +g \begin{bmatrix} -\theta_2 \\ \theta_1 + 3\theta_2 \end{bmatrix} \Rightarrow s \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & 0 & 0 \\ g & 3g & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

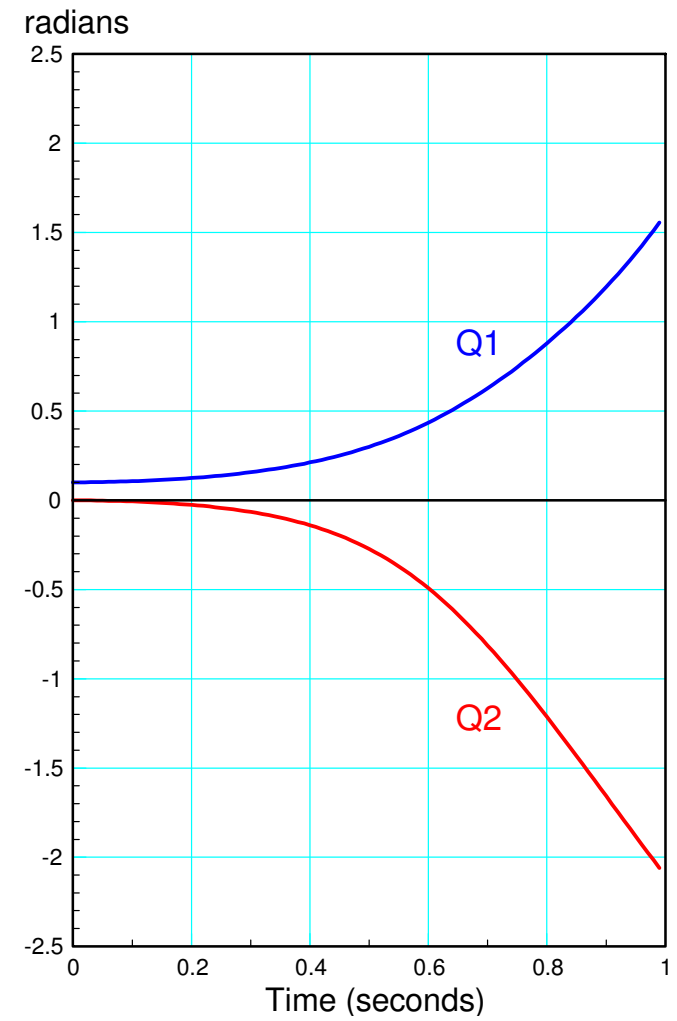
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# Free-Fall Simulation

- $g = +9.8 \text{ m/s}^2$

## Eigenvalues

- -5.0652
- +5.0652
- -1.9348
- +1.9348



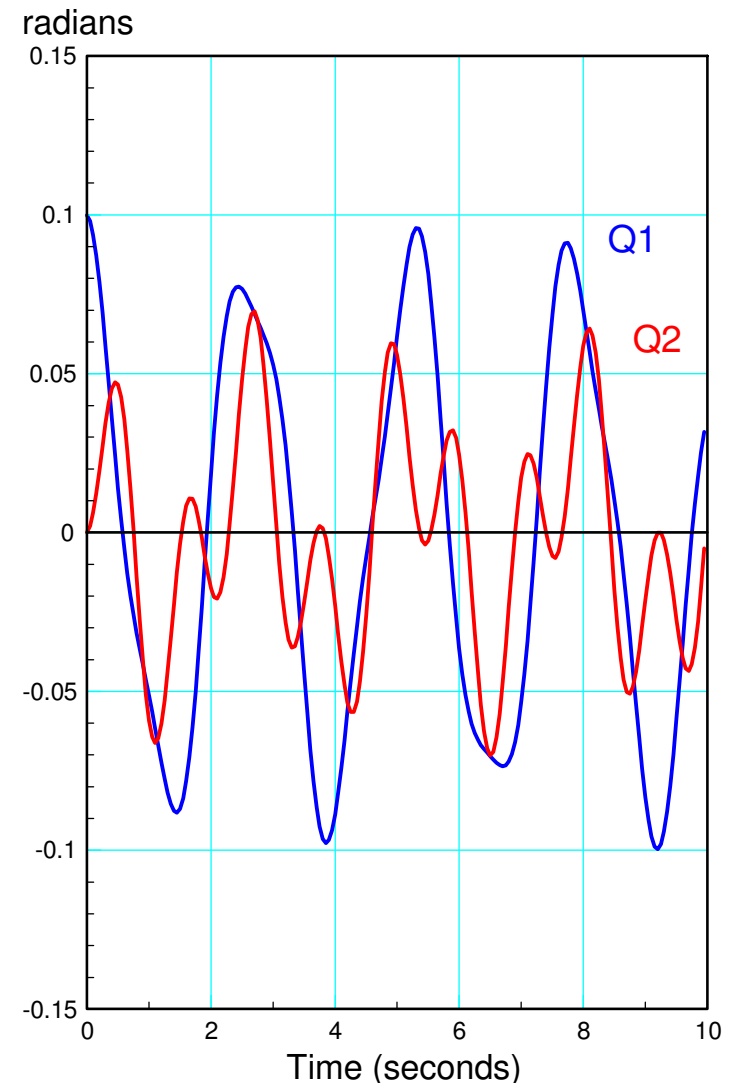
# Free Fall Simulation

- $g = -9.8 \text{ m/s}^2$

## Eigenvalues

$\text{eig}(A)$

```
0 + 5.0652i  
0 - 5.0652i  
0 + 1.9348i  
0 + 1.9348i
```



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## Summary

LaGrangians are able to determine the dynamics of a 2-link arm

$$\begin{bmatrix} (3 + 2c_2) & (1 + c_2) \\ (1 + c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g \begin{bmatrix} 2s_1 + s_{12} \\ s_{12} \end{bmatrix}$$

The same procedure is used for 3-link and 6-link robotic arms

- The derivation takes about 10 pages though.
- But, you only have to derive it once for a given robot.

Note: We spend considerable time in ECE 761 Robotics designing feedback controllers for this system.

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