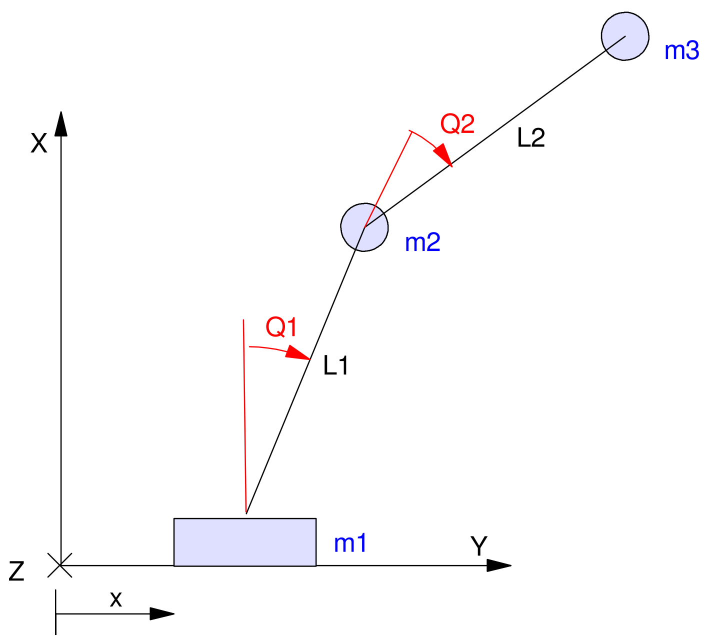
Dynamics of a Double Pendulum

A Gantry system with a second mass added to it is shown below. Assume

- m1 = m2 = m3 = 1kg
- l1 = l2 = 1m



Determine the position of each mass as a funciton of x, θ_1, and θ_2

Mass 1:

$$y_1 = x \qquad x_1 = 0$$

Mass 2:

$$y_2 = y_1 + \sin(\theta_1) \qquad x_2 = x_1 + \cos(\theta_1)$$

Mass 3:

$$y_3 = y_2 + \sin(\theta_1 + \theta_3) \qquad x_3 = x_2 + \cos(\theta_1 + \theta_2)$$

Simplifying and using shorthand notation

$$y_1 = x \qquad x_1 = 0$$

$$y_2 = x + s_1 \qquad x_2 = c_1$$

$$y_3 = x + s_1 + s_{12} \qquad x_3 = c_1 + c_{12}$$

Determine the kinetic energy of the system

Take the derivatives:

$$\dot{y}_1 = \dot{x} \qquad \dot{x}_1 = 0$$

$$\dot{y}_2 = \dot{x} + c_1\dot{\theta}_1 \qquad \dot{x}_2 = -s_1\dot{\theta}_1$$

$$\dot{y}_3 = \dot{x} + c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) \qquad \dot{x}_3 = -s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = \tfrac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \tfrac{1}{2}m_3\left(\dot{x}_3^2 + \dot{y}_3^2\right)$$

$$KE = \tfrac{1}{2}(\dot{x}^2) + \tfrac{1}{2}\left(\left(\dot{x} + c_1\dot{\theta}_1\right)^2 + \left(-s_1\dot{\theta}_1\right)^2\right)$$

$$+\tfrac{1}{2}\left(\left(\dot{x} + c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2 + \left(-s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2\right)$$

Simplifying

$$\boxed{KE = \tfrac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \tfrac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right)}$$

Determine the potential energy of the system

$$PE = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$

$$PE = gc_1 + g(c_1 + c_{12})$$

$$\boxed{PE = 2gc_1 + gc_{12}}$$

Force on the Mass #1:

Form the LaGrangian:

$$L = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - 2gc_1 - gc_{12}$$

Determine the force on the mass:

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$F = \frac{d}{dt}\left(3\dot{x} + 2c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right) - 0$$

$$F = 3\ddot{x} + 2c_1\ddot{\theta}_1 - 2s_1\dot{\theta}_1^2 + c_{12}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$$

Torque on Q1

$$L = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - 2gc_1 - gc_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$T_1 = \frac{d}{dt}\left(2\dot{\theta}_1 + \left(\dot{\theta}_1 + \dot{\theta}_2\right) + 2c_1\dot{x} + c_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_2\dot{\theta}_1 + c_{12}\dot{x}\right)$$
$$-\left(-2s_1\dot{x}\dot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + 2gs_1 + gs_{12}\right)$$

$$T_1 = 2\ddot{\theta}_1 + \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - 2s_1\dot{x}\dot{\theta}_1 + 2c_1\ddot{x} - s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_2$$
$$+c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\ddot{x}$$
$$+2s_1\dot{x}\dot{\theta}_1 + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - 2gs_1 - gs_{12}$$

Torque on Q2

$$L = \tfrac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \tfrac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - 2gc_1 - gc_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$T_2 = \tfrac{d}{dt}\left(\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_2\dot{\theta}_1 + c_{12}\dot{x}\right) - \left(-s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + gs_{12}\right)$$

$$T_2 = \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\ddot{x}$$
$$+s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - gs_{12}$$

Nonlinear Dynamics of a Double Pendulum:

$$F = 3\ddot{x} + 2c_1\ddot{\theta}_1 - 2s_1\dot{\theta}_1^2 + c_{12}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$$

$$T_1 = 2\ddot{\theta}_1 + \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - 2s_1\dot{x}\dot{\theta}_1 + 2c_1\ddot{x} - s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_2$$
$$+c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\ddot{x}$$
$$+2s_1\dot{x}\dot{\theta}_1 + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - 2gs_1 - gs_{12}$$

$$T_2 = \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\ddot{x}$$
$$+s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - gs_{12}$$

Another way to write this is

$$M\ddot{\Theta} = C\left(\theta, \dot{\theta}\right) + G(\theta) + T$$

where

- M is the mass (inertia) matrix
- C() are the coriolis forces
- G() is the gravity matrix, and
- T is the input (torques and forces).

Placing the dynamics in this form:

$$3\ddot{x} + 2c_1\ddot{\theta}_1 + c_{12}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) = F + 2s_1\dot{\theta}_1^2 + -s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$$

$$\begin{aligned} 2\ddot{\theta}_1 + \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + 2c_1\ddot{x} + c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = \quad & T_1 + 2s_1\dot{x}\dot{\theta}_1 + s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_2 \\ & + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ & -2s_1\dot{x}\dot{\theta}_1 - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + 2gs_1 + gs_{12} \end{aligned}$$

$$\begin{aligned} \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = \quad & T_2 + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ & -s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) - s_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + gs_{12} \end{aligned}$$

Placing in matrix form:

$$\begin{bmatrix} 3 & (2c_1 + c_{12}) & c_{12} \\ (2c_1 + c_{12}) & (3 + 2c_2) & (1 + c_2) \\ c_{12} & (1 + c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} +2s_1\dot{\theta}_1^2 + -s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ +2s_1\dot{x}\dot{\theta}_1 + s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_2 + s_2\dot{\theta}_1\dot{\theta}_2 - 2s_1\dot{x}\dot{\theta}_1 \\ +s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) \end{bmatrix} + g\begin{bmatrix} 0 \\ +2s_1 + s_{12} \\ s_{12} \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

Linearized Dynamics

For small angles

$$c_1 \approx 1$$

$$s_1 \approx \theta_1$$

$$\dot{\theta}_1\dot{\theta}_2 \approx 0$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g\begin{bmatrix} 0 \\ +2\theta_1 + \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

If the only input is the force on the base

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g\begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & 1 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}F$$

In State-Space form

$$s\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2g & 0 & 0 & 0 & 0 \\ 0 & 3g & g & 0 & 0 & 0 \\ 0 & -3g & 3g & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}F$$

Checking the controllability of the system:

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> A = [Z,I ; inv(M)*G*9.8,Z]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
         0   -19.6000    -0.0000          0          0          0
         0    29.4000    -9.8000          0          0          0
         0   -29.4000    29.4000          0          0          0

>> B = [0;0;0;1;-1;1];

>> rank([B,A*B,A^2*B,A^3*B,A^4*B,A^5*B])

     6
```

Surprisingly, the system is controllable from the force input.

Checking the eigenvalues:

```
>> eig(A)

         0
         0
   -6.8099
   -3.5250
    6.8099
    3.5250
```

The open-loop system is unsable as it should be. If you turn it off, it will fall.

If you reverse gravity, this is a double gantry system

```
>> A = [Z,I ; -inv(M)*G*9.8,Z]

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   19.6000    0.0000         0         0         0
         0  -29.4000    9.8000         0         0         0
         0   29.4000  -29.4000         0         0         0

>> B = [0;0;0;1;-1;1]

     0
     0
     0
     1
    -1
     1

>> eig(A)

        0
        0
   0.0000 + 6.8099i
   0.0000 - 6.8099i
  -0.0000 + 3.5250i
  -0.0000 - 3.5250i

>> rank([B,A*B,A^2*B,A^3*B,A^4*B,A^5*B])

ans =      6
```

This too is controllable from the force input.