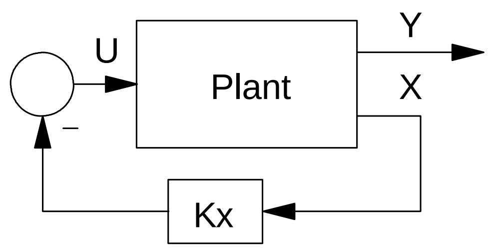
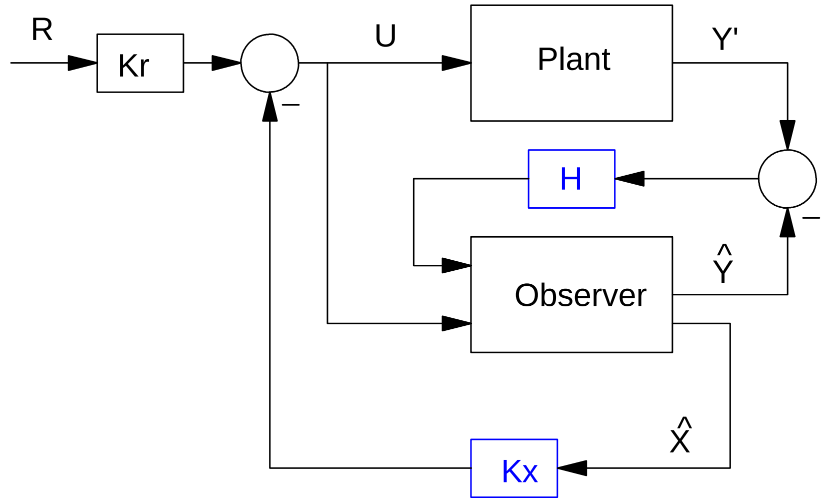
Separation Principle & Full-Order Observer Design

Suppose you want to design a feedback controller. Using full-state feedback you can place the poles of the closed-loop system at will.



If the states are measurable, the full-state feedback gains, Kx, place the poles of the closed-loop system.

If the system states are not measured, however, you can use an observer to estimate these states. These estimates can then be used to compute the input, U



If the states are not measurable, an observer is used to estimate the states.

This leads to a problem, however. How to you desgin both the observer gain, H, as well as the controller gains, Kx at the same time?

Separation Principle:

Assume you have a plant

$$sX = AX + BU$$

$$Y = CX$$

along with a state-estimator

$$s\hat{X} = A\hat{X} + BU + H\left(Y - \hat{Y}\right)$$

$$\hat{Y} = C\hat{X}$$

along with the control law

$$U = K_r R - K_x \hat{X}$$

The augmented system becomes

$$s\begin{bmatrix} X \\ \hat{X} \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & \hat{A} - HC \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}U$$

or, substituting for U

$$s\begin{bmatrix} X \\ \hat{X} \end{bmatrix} = \begin{bmatrix} A & -BK_x \\ HC & \hat{A} - HC - BK_x \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}K_r R$$

Do a change of variable:

$$\begin{bmatrix} X \\ E \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix}$$

Do a similarity transform

$$s\begin{bmatrix} X \\ E \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A & -BK_x \\ HC & \hat{A} - HC - BK_x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X \\ E \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix}K_r R$$

Simplify

$$s\begin{bmatrix} X \\ E \end{bmatrix} = \begin{bmatrix} A - BK_x & BK_x \\ 0 & A - HC \end{bmatrix}\begin{bmatrix} X \\ E \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}K_r R$$

Note that the augmented system's A matrix is diagonal. This means that the eigenvalues of the sysetm are the eigenvalues of each diagonal element.

- The eigenvalues of the closed-loop system are the eigenvalues of (A - BKx) and (A - HC) combined
- The feedback controller gain, Kx, and the observer gain, H, have no effect on each other.

This is the separation principle:

You can design the full-state feedback gains without any regard of the observer gains and visa versa.

Selection of Observer Gains

Where you place the poles of (A - BKx) is determined by the design requirements. These determine how the closed-loop system will behave. But, where should you place the poles of (A - HC)?

There are a couple of ways to do this. One looks at how much noise is in the measurements of the input (U) and output (Y). This leads to a Kalman Filter - which we will cover later. A second way is presented here.

Since the full-state feedback gains depend upon good accurate measurements of the states, you could argue that the observer should be fast relative to the closed-loop system so that the state estimates converge quickly, giving you good measurements for the feedback gains U = -Kx X. You don't want to make the observer too fast, however, since a quic response results in large gains, which can cause numerical problems and noise issues. A reasonable compromise is thus:

Choose the observer gains so that they converge 3 to 10 times faster than the closed-loop system.

Example 1

Design a fedback controller for the following system using only the output measurement, Y, so that the closed-loop system has

- No error for a step input,
- A 2% settling time of 4 seconds, and
- A DC gain of one.

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

Solution: To track a constant set-point, design a servo compensator with a pole at s = 0. The augmented system then becomes

$$\begin{bmatrix} sX \\ \cdots \\ sZ \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & \vdots & 0 \\ 1 & -2 & 1 & 0 & \vdots & 0 \\ 0 & 1 & -2 & 1 & \vdots & 0 \\ 0 & 0 & 1 & -1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \begin{bmatrix} X \\ \cdots \\ Z \end{bmatrix} + \begin{bmatrix} B \\ \cdots \\ 0 \end{bmatrix} U$$

To meet the design specs, use full-state feedback to place the closed-loop poles at {-1, -2, -3, -4, -5}

```
-->A5 = [A,zeros(4,1);C,0]

  - 2.    1.    0.    0.    0.
    1.  - 2.    1.    0.    0.
    0.    1.  - 2.    1.    0.
    0.    0.    1.  - 1.    0.
    0.    0.    0.    1.    0.
```

```
-->B5 = [1;0;0;0;0]

    1.
    0.
    0.
    0.
    0.

-->K5 = ppl(A5,B5, [-1,-2,-3,-4,-5])

    8.    30.    77.    158.    120.
```

Next, since all four states are not measured, design an observer to estimate the states. Choose the observer poles to be 3 to 10 times faster than the plant. Let the observer poles be {-3, -4, -5, -6}

```
-->H = ppl(A', C', [-3, -4, -5, -6])'

    61.
    70.
    38.
    11.
```

So, the augmented system becomes

$$\begin{bmatrix} sX \\ sZ \\ s\hat{X} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B_zC & A_z & 0 \\ HC & 0 & A-HC \end{bmatrix} \begin{bmatrix} X \\ Z \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ B \end{bmatrix} U + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} R$$

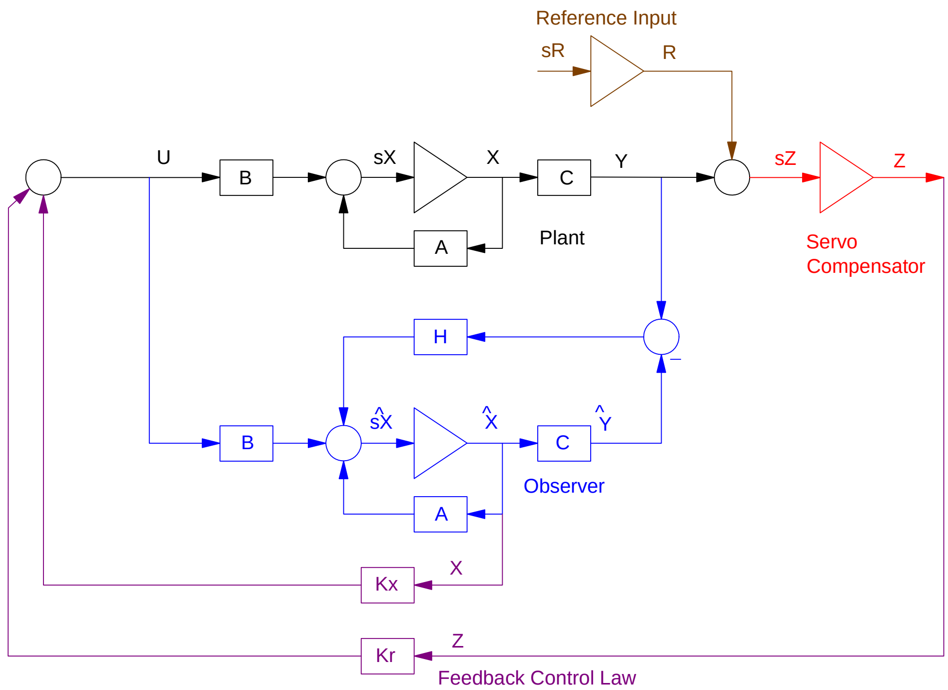
Replacing U with

$$U = \begin{bmatrix} 0 & -K_z & -K_x \end{bmatrix} \begin{bmatrix} X \\ Z \\ \hat{X} \end{bmatrix}$$

results in

$$\begin{bmatrix} sX \\ sZ \\ s\hat{X} \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ B_zC & A_z & 0 \\ HC & -BK_z & A-HC-BK_x \end{bmatrix} \begin{bmatrix} X \\ Z \\ \hat{X} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} R$$

Use the Matlab command Step2 to input an initial condition for a step input.



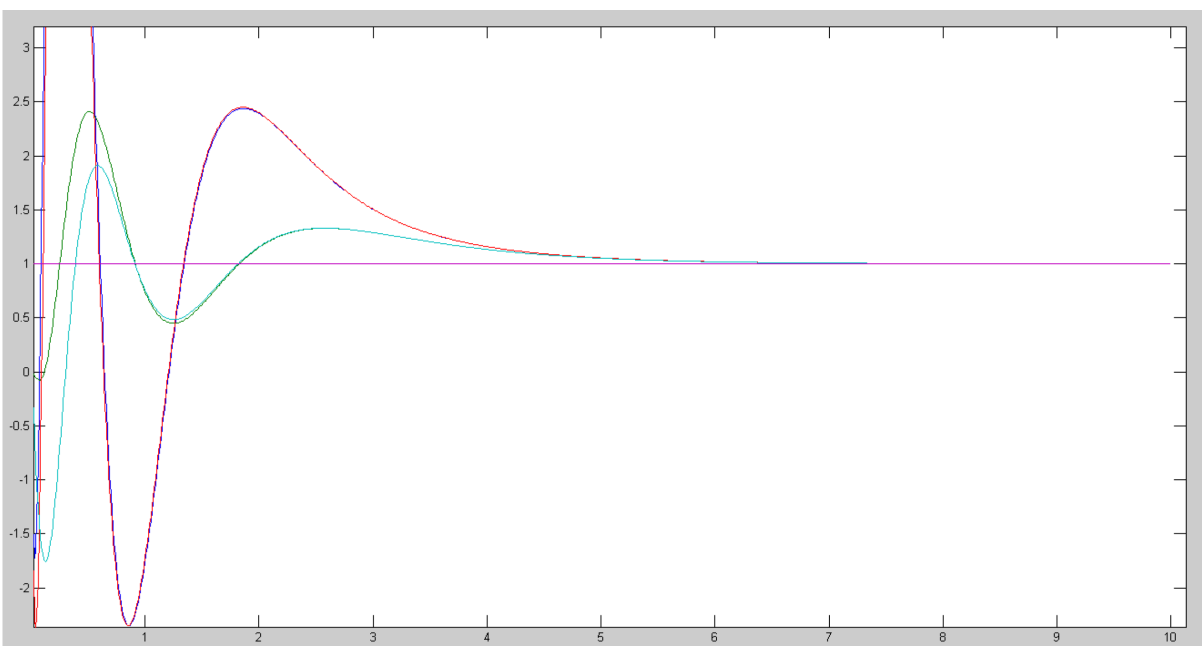
Net System: Plant + Observer + Servo Compensator + Constant Reference Input

```
>> A9 = [  A, -B*Kz,   -B*Kx     ;
            C,    0,      C*0     ;
          H*C, -B*Kz, A-H*C-B*Kx ;
          0*C,    0,      0*C    ]

          plant                 servo        observer
   -2      1      0      0 :  -120 :    -8    -30    -77   -158
    1     -2      1      0 :     0 :     0      0      0      0
    0      1     -2      1 :     0 :     0      0      0      0
    0      0      1     -1 :     0 :     0      0      0      0
   - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
    0      0      0      1 :     0 :     0      0      0      0
   - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
    0      0      0     61 :  -120 ;   -10    -29    -77   -219
    0      0      0     70 :     0 :     1     -2      1    -70
    0      0      0     38 :     0 ;     0      1     -2    -37
    0      0      0     11 :     0 ;     0      0      1    -12
```

```
>> X0 = [0 0 0 0 :  0 : 0.2 0.3 0.4 0.5]'

         0
         0   plant initial conditions
         0
         0
    - - - -
         0   servo initial condition
    - - - -
    0.2000
    0.3000
    0.4000   observer initial conditions (error)
    0.5000
```

This is a really noisy plot - so just plot angle, position, and their estimates:

```
>> C9 = [ 1 0 0 0 0 0 0 0 0 ;           % position
          0 1 0 0 0 0 0 0 0 ;           % angle
          0 0 0 0 0 1 0 0 0 ;           % position estimate
          0 0 0 0 0 0 1 0 0 ];          % angle estimate

>> D9 = zeros(4,1);

>> y = step2(A9, B9, C9, D9, X0, t);
>> plot(t,y);
```



Note that

- If the error in the state estimator (observer) is initially zero (red line), the system behaves just like it did when the states were used to compute U.

- If the error in the state estimator is non-zero (blue line), the error is quickly driven to zero (pole at s = -3). Once driven to zero, the closed-loop system starts to behave correctly.

Also note that the separation principle holds: the eigenvalues of the closed-loop system are

- The eigenvalues of (A - BKx). and
- The eigenvalues of (A - HC)

```
>> eig(A9)

  -1.0000      poles of (A - B Kx) are { -1, -2, -3, -4, -5 }
  -2.0000
  -3.0000
  -4.0000
  -5.0000

  -3.0000      poles of ( A - HC ) are { -3, -4, -5, -6 }
  -4.0000
  -5.0000
  -6.0000
```

Example 2:

In the previous simulation, when the observer had a poor initial estimate of the states (blue line), the poor estimates produced wrong inputs (U) which made the plant behave badly.

How do you adjust the feedback gains, Kx, so that they only kick in once the observer converges?

Solution: This is a *very* difficult problem and has yet to be solved. You're trying to adjust the feedback gains on-the-fly in a way that the system remains stable while the state-estimator tries to figure out what the plant is doing.