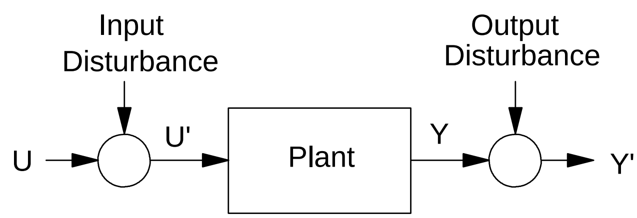
Observers and Disturbance Rejection

Suppose a plant has a disturbance at the input or output (sensor). How do you estimate the states in the presence of such disturbances?



If you ignore the disturbances, this will throw off the state estimates. Bad estimates can adversely affect the feedback control system.

For example, consider the 4-state heat equation with an input disturbance:

$$sX = AX + B(U + d)$$

$$Y = CX$$

where 'd' is a constant disturbance at the input. If you ignore this disturbance and build an observer as

$$s\hat{X} = A\hat{X} + BU + H\left(Y - \hat{Y}\right)$$

$$\hat{Y} = C\hat{X}$$

The error in the state estimate is:

$$E = X - \hat{X}$$

Taking the derivative:

$$sE = sX - s\hat{X}$$

Substituting:

$$sE = (AX + B(U + d)) - \left(A\hat{X} + BU + H\left(Y - \hat{Y}\right)\right)$$

With some algebra:

$$sE = A\left(X - \hat{X}\right) + Bd - HC\left(X - \hat{X}\right)$$

or

$$sE = (A - HC)E + Bd$$

Assuming a step input, meaning all the states are constand (sE = 0), then

$$E = -(A - HC)^{-1}Bd$$

The disturbance will create an error in the state estimates. Bad estimates can result in a badly performing feedback system.

This is also a problem if you have an output disturbance. In this case, the system becomes

$$sX = AX + BU$$

$$Y = CX + d$$

If the observer ignores this disturbance, then

$$s\hat{X} = A\hat{X} + BU + H\left(Y - \hat{Y}\right)$$

$$\hat{Y} = C\hat{X}$$

The error is

$$E = X - \hat{X}$$

Taking the derivative:

$$sE = sX - s\hat{X}$$

Substituting:

$$sE = (AX + BU) - \left(A\hat{X} + BU + H\left(Y - \hat{Y}\right)\right)$$

Rewriting this:

$$sE = (AX + BU) - \left(A\hat{X} + BU + H\left(CX + d - C\hat{X}\right)\right)$$

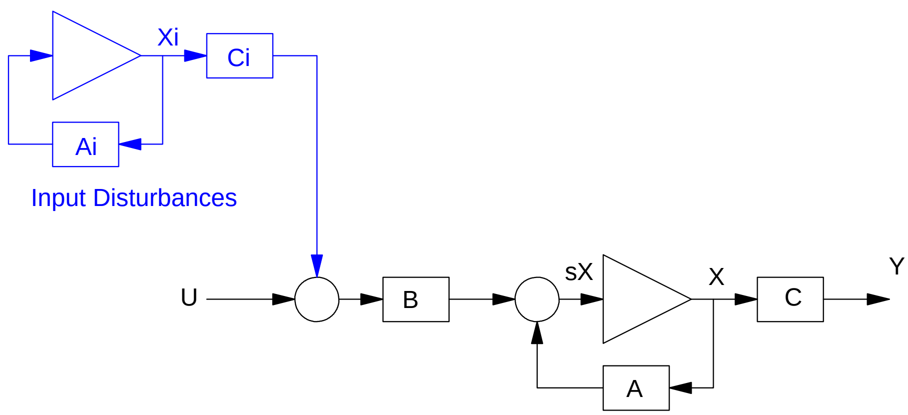
$$sE = (A - HC)E - Hd$$

Again, the disturbance affects the error in the state estimates. Bad estimates can result in poorly behaving systems.

If a system has a disturbance, it needs to be taken into account with the observer.

Case 1: Input Disturbances

Consider first the case where the input has a disturbance (meaning the actual input to the plant has a component you didn't command with U - such as a constant offset).



Augmented System (A5, B5, C5) for a Plant with an Input Disturbance

In state-space, you can model the augmented system (plant plus disturbance) as:

$$s\begin{bmatrix} X \\ X_i \end{bmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}\begin{bmatrix} X \\ X_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}U$$

$$Y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} X \\ X_d \end{bmatrix}$$

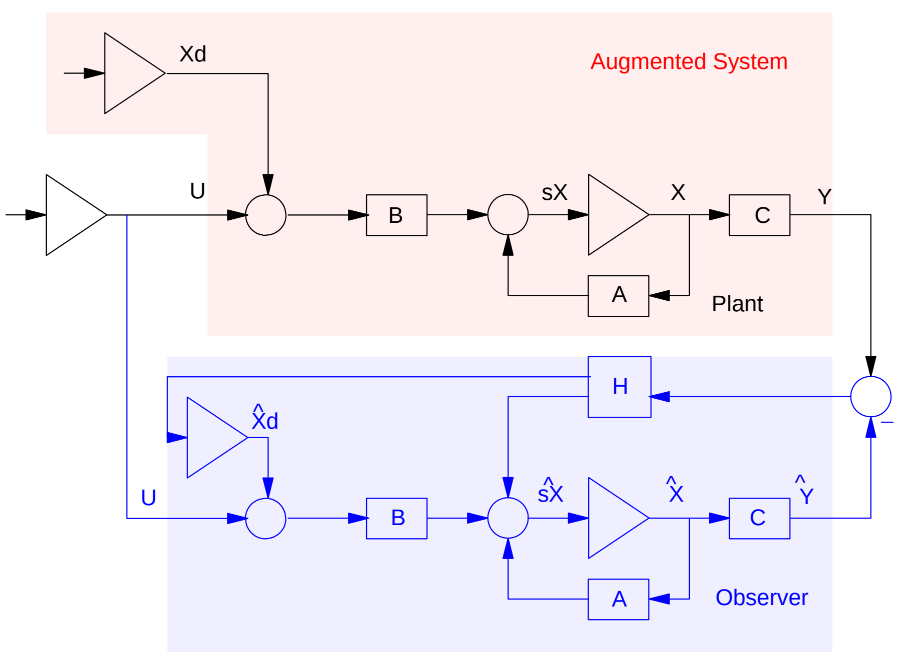
You can then build a full-order observer for the plant plus disturbance. This will result in correct estimates of the plant states, with the effect of the offset removed.

Example: 4th-Order heat equation with a constant offset at the input.

First, create the augmented system:

```
-->A5 = [A,B;0*C,0]

 - 2.     1.     0.     0. :   1.
   1.   - 2.     1.     0. :   0.
   0.     1.   - 2.     1. :   0.
   0.     0.     1.   - 1. :   0.
  - - - - - - - - - - - - - - -
   0.     0.     0.     0. :   0.

>> B5 = [B; 0]

     1
     0
     0
     0
 - - - -
     0


-->C5 = [C,0]

   0.     0.     0.     1. :   0.
```

Second, design a full-order observer for the augmented system



Using Bass-Gura, you can place the poles at will. Placing the observer poles at {-3, -3, -3, -3, -3} somewhat arbitrarily results in

```
>> H5 = ppl(A5', C5', [-3,-3,-3,-3,-3])'

  147.0000
   72.0000
   27.0000
    8.0000
  243.0000

>> eig(A5-H5*C5)

  -3.0047
  -3.0014 + 0.0044i
  -3.0014 - 0.0044i
  -2.9962 + 0.0027i
  -2.9962 - 0.0027i
```

To simulate for a step input, add a state for U:

$$\begin{bmatrix} sX_5 \\ s\hat{X}_5 \end{bmatrix} = \begin{bmatrix} A_5 & 0 \\ H_5C_5 & A_5 - H_5C_5 \end{bmatrix} \begin{bmatrix} X_5 \\ \hat{X}_5 \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} U$$

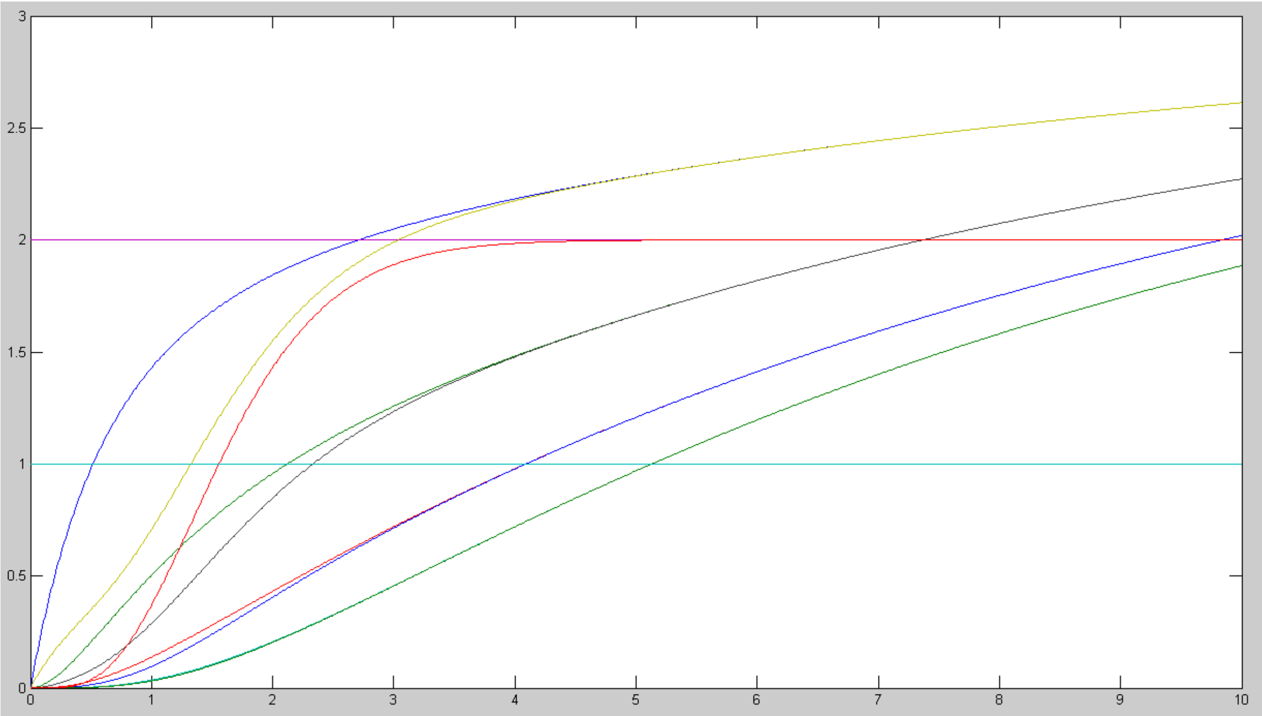
Simulate this in Matlab using initial conditions on the disturbance (d = 2) and the input (U = 1). Note that this is now an 11th-order system

- 4 states for the plant
- 1 for the disturbance
- 5 for the observer (plant plus disturbance)

```
>> A10 = [   A5,   zeros(5,5),  B5 ;
           H5*C5, A5-H5*C5,    B5 ]

                X      y     d         est(X)      ye  ext(d)
  -2    1     0     0 :   1 :   0     0     0     0 :   0
   1   -2     1     0 :   0 ;   0     0     0     0 :   0
   0    1    -2     1 :   0 ;   0     0     0     0 :   0
   0    0     1    -1 :   0 ;   0     0     0     0 :   0
 - - - - - - - - - - - - - - - - - - - - - - - - - - 
   0    0     0     0 :   0 :   0     0     0     0 :   0
- - - - - - - - - - - - - - - - - - - - - - - - - - 
   0    0     0   147 :   0 :  -2     1     0  -147 :   1
   0    0     0    72 :   0 :   1    -2     1   -72 :   0
   0    0     0    27 :   0 :   0     1    -2   -26 :   0
   0    0     0     8 :   0 :   0     0     1    -9 :   0
- - - - - - - - - - - - - - - - - - - - - - - - - - 
   0    0     0   243 :   0 :   0     0     0  -243 :   0
```

The initial conditions are:

```
>> X0 = [0;0;0;0;2; 0;0;0;0;0 ]

     0
     0    initial states on the plant is zero
     0
     0
   - - -
     2    disturbance is 2
   - - -
     0
     0    initial value of observer
     0
     0
   - - -
     0   initial value of estimate of d


>> C10 = eye(10,10);
>> D10 = zeros(10,1);

>> y = step2(A10, B10, C10, D10, X0, t);
>> plot(t,y);
```



Plant and Observer states with an Input Disturbance of 2.
The state estimates converge to their correct values in about 4 seconds

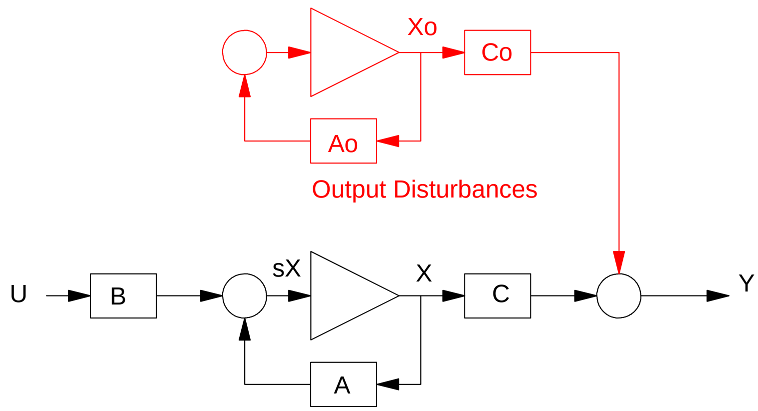
Note that

- The disturbance estimates converges in about 4 seconds. This is expected since the observer poles were placed at {-3, -3, -3, -3, -3}
- The disturbance is a constant, 2 (pink line)

- The estimate of the disturbance converges to 2 in about 4 seconds (red line)
- All state estimates converge to the actual states in spite of the constant output disturbance.

Case 2: Output Disturbance

Next, consider the case where the output (sensor) has a disturbance:



Augmented System (A5, B5, C5) includes the plant and an output disturbance.

In state-space, the augmented system (plant plus disturbance) is:

$$s\begin{bmatrix} X \\ X_o \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_o \end{bmatrix}\begin{bmatrix} X \\ X_o \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}U$$

$$Y = \begin{bmatrix} C & C_o \end{bmatrix}\begin{bmatrix} X \\ X_o \end{bmatrix}$$

If the plant and disturbance is observable, you can design a full-order observer to estimate all of the states. This likewise allows you to remove the effect of the disturbance on the state estimates.

Example: Suppose the sensor has a constant offset for the 4th-order heat equation. To model a constant disturbance, let

$$sX_o = [0]X_0$$

$$Y_0 = [1]X_o$$

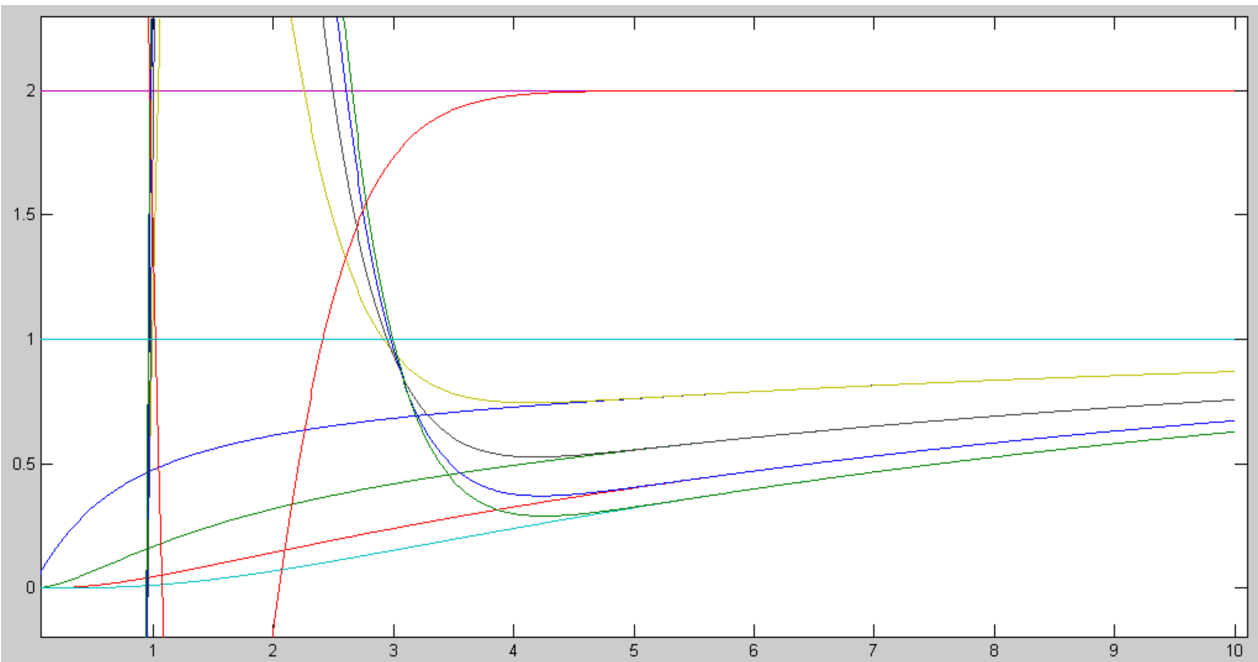
Creating the augmented system

```
-->A5 = [A,B*0;0*C,0]

   -2    1    0    0 :   0
    1   -2    1    0 :   0
    0    1   -2    1 :   0
    0    0    1   -1 :   0
   - - - - - - - - - - - - -
    0    0    0    0 :   0

>>B5 = [B ; 0];

    1
    0
    0
    0
   - -
    0

>>C5 = [0,0,0,1,1]

    0    0    0    1 :   1
```

Using Bass-Gura, you can then place the poles at will

```
>> H5 = ppl(A5', C5', [-3, -3, -3, -3, -3])'

H5 =

  -96.0000
 -171.0000
 -216.0000
 -235.0000
  243.0000

>> eig(A5-H5*C5)

ans =

  -2.9863
  -2.9958 + 0.0130i
  -2.9958 - 0.0130i
  -3.0111 + 0.0080i
  -3.0111 - 0.0080i

>> A10 = [A5, zeros(5,5);    H5*C5, A5-H5*C5]

                X    y     d          ext(X )      ext(d)
 -2    1    0    0 :   0 :   0    0    0    0 :   0
  1   -2    1    0 :   0 :   0    0    0    0 :   0
  0    1   -2    1 :   0 :   0    0    0    0 :   0
  0    0    1   -1 :   0 :   0    0    0    0 :   0
 - - - - - - - - - - - - - - - - - - - - - - - - - -
  0    0    0    0 :   0 :   0    0    0    0 :   0
 - - - - - - - - - - - - - - - - - - - - - - - - - -
  0    0    0  -96 : -96 :  -2    1    0   96 :  96
  0    0    0 -171 :-171 :   1   -2    1  171 : 171
  0    0    0 -216 :-216 :   0    1   -2  217 : 216
  0    0    0 -235 :-235 :   0    0    1  234 : 235
 - - - - - - - - - - - - - - - - - - - - - - - - - -
  0    0    0  243 : 243 :   0    0    0 -243 :-243
```

```
>> X0 = [0;0;0;0;2; 0;0;0;0;0]

     0
     0         initial value of plant states
     0
     0
   - - -
     2         output disturbance = 2
   - - -
     0
     0         initial value of observer states
     0
     0
   - - -
     0         initial value of estimate of the disturbance

>> B10 = [B5 ; B5];
>> C10 = eye(10,10);

>> y = step2(A10, B10, C10, D10, X0, t);
>> plot(t,y);
```



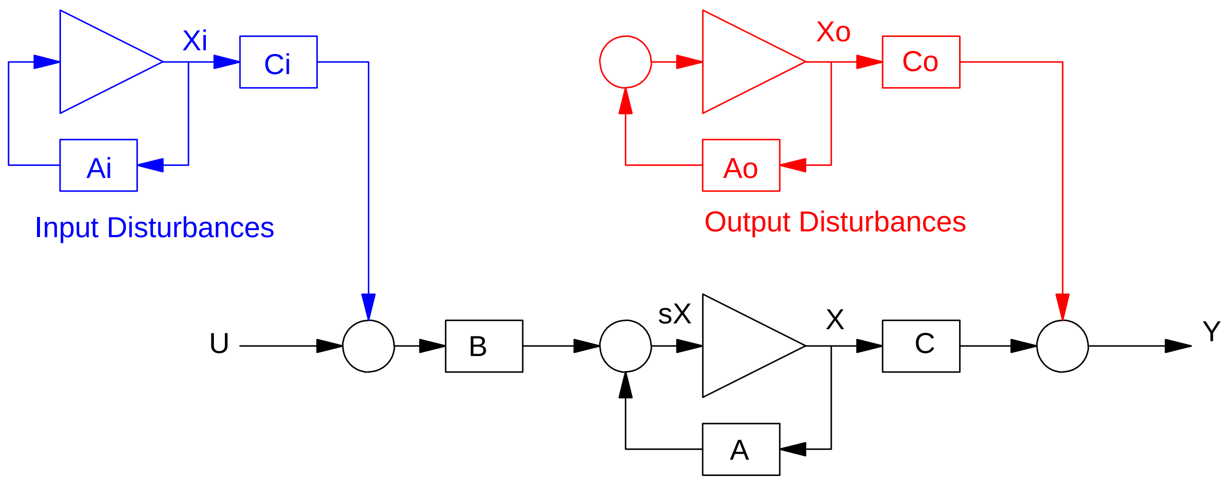
Plant States and the Observer Estimates for a Step Input (U = 1) and an output disturbance of 2 (d = 2)

Note that

- The disturbance estimates converges in about 4.3 seconds. This is expected since the observer poles were placed at {-3, -3, -3, -3, -3}
- The disturbance is a constant, 2 (pink line)
- The estimate of the disturbance converges to 2 (red line) in about 4.3 seconds
- All states converge in spite of the constant output disturbance.

Case 3: Input and Output Disturbances

Finally, consider the case where the system has both input and output disturbances:



In state-space, the augmented system becomes:

$$s\begin{bmatrix} X \\ X_i \\ X_o \end{bmatrix} = \begin{bmatrix} A & BC_i & 0 \\ 0 & A_i & 0 \\ 0 & 0 & A_o \end{bmatrix}\begin{bmatrix} X \\ X_i \\ X_o \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} C & 0 & C_o \end{bmatrix}\begin{bmatrix} X \\ X_i \\ X_o \end{bmatrix}$$

Note that there is a problem here. If you check the observability of the system using a PBH test

$$\rho\left(\begin{bmatrix} C \\ A - \lambda I \end{bmatrix}\right) = N$$

$$\rho\left(\begin{bmatrix} A-\lambda I & BC_i & 0 \\ 0 & A_i - \lambda I & 0 \\ 0 & 0 & A_o - \lambda I \\ \cdots & \cdots & \cdots \\ C & 0 & C_o \end{bmatrix}\right) = N$$

you can see that matrix is not full-rank (several rows are zero) if Ai and Ao share eigenvalues.

This makes sense. For example, suppose you have a constant disturbance. This means the output will have a constant offset. If you have both input and output disturbances, you don't know how much of this offset is produced by each term.