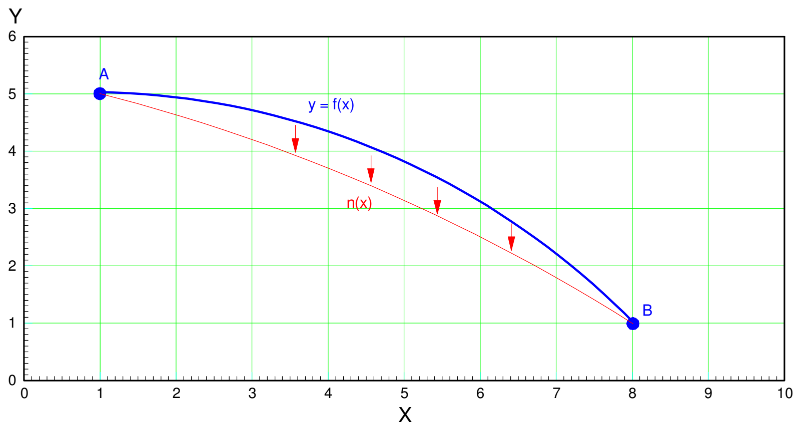
Calculus of Variations

Calculus of Variations is a branch of mathematics dealing with optimizing functionals. A functional is a function of functions. For example

$$J(x) = \int_a^b F(t, x, \dot{x})dt$$

computes a cost, J, for a fuven function x(t). For different x(t)'s, you'll have different costs. The problem of finding x(t) which minimizes (or maximizes) J is generally the problem solved with Calculus of Varations.



Problem in Calculus of Variations: Find the function, y = f(x), which minimizes a cost function.
Any pertubation (n(x)) will result in a higher cost

Euler Legrange Equation with One Dependent Variable

The minimum is found from

$$\frac{dJ}{d\xi} = \lim_{\xi \to 0} \left(\frac{J(x+\xi n) - J(x)}{\xi} \right) = 0$$

Expanding using a Taylor's series (where h.o.t. means "higher order terms")

$$F(t, x+\xi n, \dot{x}+\xi \dot{n}) = F(t, x, \dot{x}) + \xi \frac{dF}{d\xi} + h.o.t.$$

$$= F(t, x, \dot{x}) + \xi (F_x n + F_{\dot{x}} \dot{n}) + h.o.t.$$

The partial is then

$$\frac{dJ}{d\xi} = \lim_{\xi \to 0} \left(\frac{\int_a^b (F(t.x+\xi n, \dot{x}+\xi \dot{n}) - F(t, x, \dot{x}))dx}{\xi} \right)$$

$$\frac{dJ}{d\xi} = \lim_{\xi \to 0} \left(\frac{\xi \left(\int_a^b (F_x n + F_{\dot{x}} \dot{n})dt \right) + h.o.t.}{\xi} \right)$$

Taking the limit results in

$$\frac{dJ}{d\xi} = \int_a^b (F_x n + F_{\dot{x}} \dot{n}) dt$$

Note that

$$F_{\dot{x}} \dot{n} = \frac{d}{dx}(F_{\dot{x}} n) - \frac{d}{dx}(F_{\dot{x}}) n$$

which allows you to writh this integral as

$$\frac{dJ}{d\xi} = \int_a^b \left(\left(F_x + \frac{d}{dt}(F_{\dot{x}}) \right) n + \frac{d}{dt}(F_{\dot{x}} n) \right) dt$$

or

$$\frac{dJ}{d\xi} = \int_a^b \left(F_x + \frac{d}{dt}(F_{\dot{x}}) \right) n \cdot dt + (F_{\dot{x}} n) \Big|_a^b = 0$$

Since n(t) is an arbitrary function, this can only be true if

	$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$	**Euler Legrange Equation** F(t, x, dx/dt)
	$F_y - \frac{d}{dx}\left(F_{y'} \right) = 0$	F(x, y, dy/dx)

If a function x(t) minimizes a cost function, J, then x(t) must satisfy the Euler Legrange equation.

The boundary donditions are then set by the second term:

- If x(a) is fixed, then n(a) = 0
- if x(a) is free, then $F_{\dot{x}} = 0$

Example 1: Shortest Distance Between Two Points:

Find the function, y(x), which minimizes the distance between (1,5) and (8,1)

- What is the minimal cost road connecting these two points assuming uniform cost per mile

$$J = \int_a^b \left(\sqrt{dx^2 + dy^2} \right)$$

$$J = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx$$

$$F = \sqrt{1 + (y')^2}$$

The Euler Legrange equation is

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

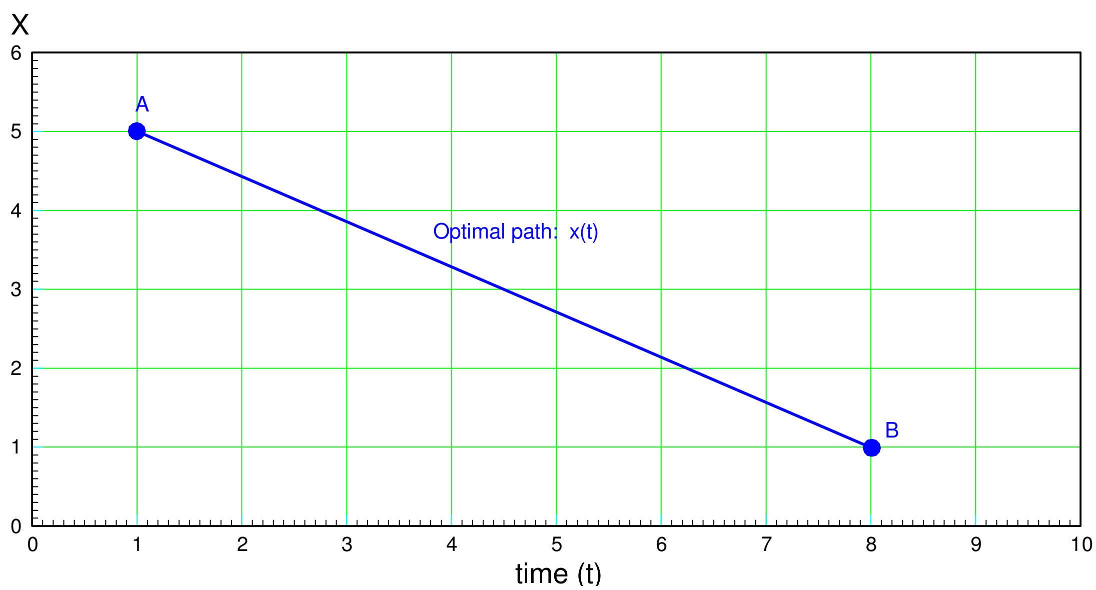
$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\frac{y'}{\sqrt{1+y'^2}} = c$$

$$y' = a$$

$$y = ax + b$$

A straight line is the shortest distance between two points



Example 2: Soap Film:

Minimize the surface area of a soap film that connects two rings centered on the y-axis:

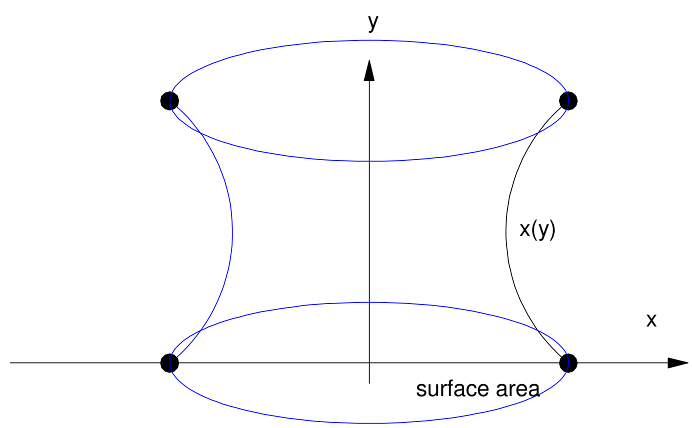
$$F = x\sqrt{1+y'^{\,2}}$$

The rings have a radius of 1 and are 1m apart:

$y(0) = 5;$ $y(5) = 5$

Alternate Problem:

- What road connects the points (0, 5) and (5,5) assuming the cost per mile is proportional to X?



J = surface area of a soap film (rotated about the y axis)

Throwing it into the Euler LeGrange equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

Since there is no 'y', this simplifies to

$$-\frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^{\,2}}}\right) = 0$$

$$\frac{xy'}{\sqrt{1+y'^{\,2}}} = a$$

Solve for y'

$$y' = \frac{a}{\sqrt{x^2-a^2}}$$

$$dy = \frac{a}{\sqrt{x^2-a^2}}dx$$

$$\int dy = \int \left(\frac{a}{\sqrt{x^2 - a^2}} \right) dx$$

To solve, do a change of variable. From trig:

$$\cosh^2 - 1 = \sinh^2$$

Rewrite as

$$\int dy = \int \frac{1}{\sqrt{\left(\frac{x}{a}\right)^2 - 1}} dx$$

Let

$$\frac{x}{a} = \cosh(\theta)$$

then

$$dx = a \sinh(\theta) \cdot d\theta$$

Plugging in

$$\int dy = \int \left(\frac{1}{\sqrt{\cosh^2(\theta) - 1}} \cdot (a \sinh(\theta)) \right) \cdot d\theta$$

$$y = \int \left(\frac{a \sinh(\theta)}{\sinh(\theta)} \right) \cdot d\theta$$

$$y = a\theta + b$$

Resubstituting for x

$$y = a\left(\cosh^{-1}\left(\frac{x}{a}\right)\right) + b$$

$$x = a \cosh\left(\frac{y-b}{a}\right)$$

cosh(x) = cosh(-x), so this is also

$$\boxed{x = a \cosh\left(\frac{y-b}{a}\right)}$$

Shape of a soap film for rings circling the y-axis

With a change of variable you this is also the solution for soap film between rings around the x-axis

$$\boxed{y = a \cosh\left(\frac{x-b}{a}\right)}$$

Shape of a soap film for rings circling the x-axis

Soap Film with Fixed End Points:

Find the shape of a soap film which rotates about the X axis. The endpoint constraints are:

$y(0) = 5$ left endpoint

$y(5) = 5$ right endpoint

A soap film rotated about the X axis is:

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

Left Endpoint:

$$5 = a\cosh\left(\frac{0-b}{a}\right)$$

Right Endpoint:

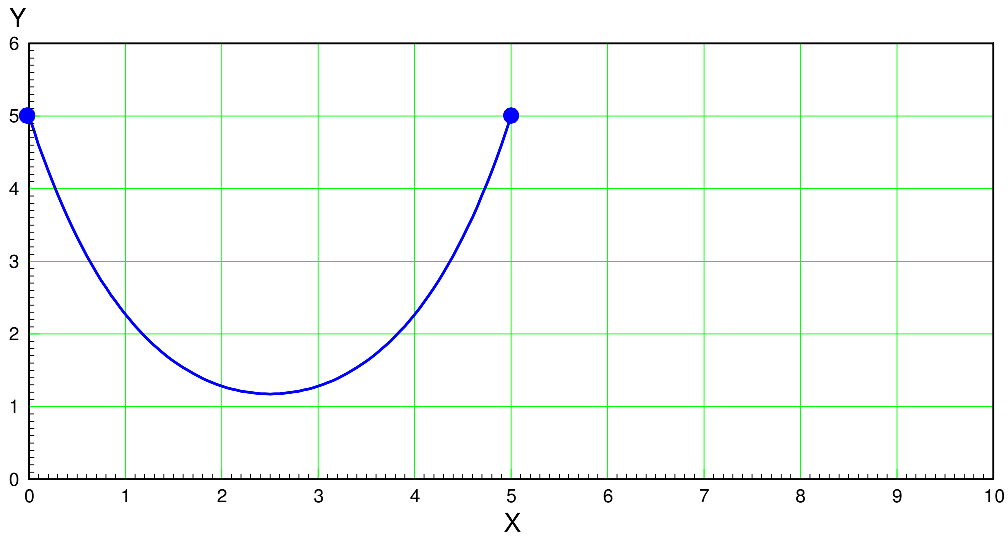
$$5 = a\cosh\left(\frac{5-b}{a}\right)$$

Solving two equations for two unknowns:

```
function y = cost(z)

   a = z(1);
   b = z(2);

   e1 = a*cosh( -b/a) - 1;
   e2 = a*cosh( (1-b)/a) - 1;

   y = e1^2 + e2^2;

   end


-->[m,n] = fminsearch('cost',[1,2])

 n  =

    1.1754749    2.5
 m  =

    1.383D-29
```

so

$$\boxed{y = 1.175\cosh\left(\frac{x-2.5}{1.175}\right)}$$



Shape of a soap film between (0,5) and (5,5) rotated about the X axis
Path of a road that minimizes the cost between (0, 5) and (5,5) assuming cost per mile is proportional to x

Soap Film with a Free Endpoint:

If the right endpoint is free, then the constraint is that

$$F_{y'} = 0$$

Example:

- Left Endpoint: x = 0, y = 5
- Right Endpoint: x = 2, y = free

Equation #1

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

$$5 = a\cosh\left(\frac{-b}{a}\right)$$

Equation #2:

$$F_{y'} = 0$$

$$F = x\sqrt{1+y'^2}$$

$$F_{y'} = \left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0$$

$$y' = 0$$

$$y = a\cosh\left(\frac{x-b}{a}\right)$$

$$y' = -a\sinh\left(\frac{x-b}{a}\right) = 0$$

at x = 2

$$y' = -a\sinh\left(\frac{2-b}{a}\right) = 0$$

b = 2

So

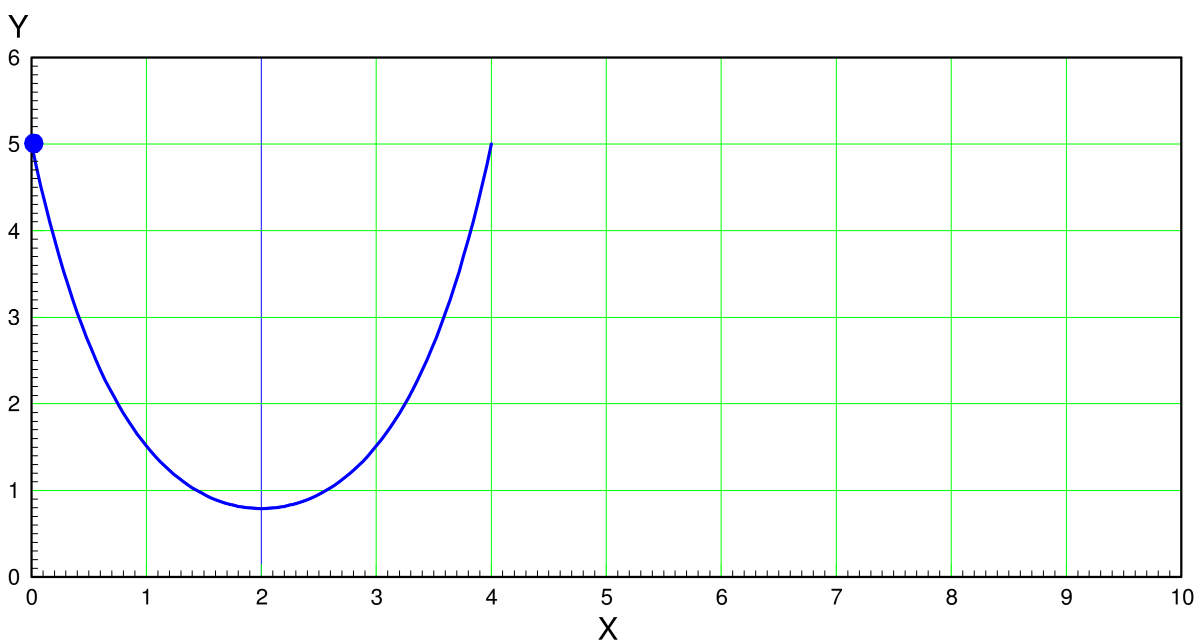
$$5 = a\cosh\left(\frac{2}{a}\right)$$

a = 0.7998

and

$$\boxed{y = 0.7898\cosh\left(\frac{x-2}{0.7898}\right)}$$

(graph shown to the right). Note that a the right endpoint (the free endpoint), y(x) is perpindicular to the surface



Path of a soap film from (0, 5) to a surface at x = 2
Path of a road that minimizes the cost when traveling from (0,5) to (2,y) assuming cost per mile is proportional to x

Euler Legrange Equation with Two Dependent Variables

If you have *two* dependent variables:

$$J = \int_a^b F(t, x, \dot{x}, u, \dot{u}) dt$$

you have Itwo Euler Legrange equations to solve

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$F_u - \frac{d}{dt}(F_{\dot{u}}) = 0$$

Euler Legrange Equation with Contraints:

Finally, if you have constraints, such as

$$G(t, x, \dot{x}, u, \dot{u}) = 0$$

you can modify the const functional by adding a Legrange multiplier, M:

$$J = \int_a^b (F(t, x, \dot{x}, u, \dot{u}) + M \cdot G(t, x, \dot{x}, u, \dot{u}))dt$$

You can then solve this functional by plugging in the boundary conditions and the constraint on G(t,x,x').

Example 3: Hanging Chain

Find the shape of a chain of length 2 hanging between two points, 1 meter apart. Assume gravity is in the -y direction (makes solving easier since there is no 'y' in the Euler LeGrange equations)

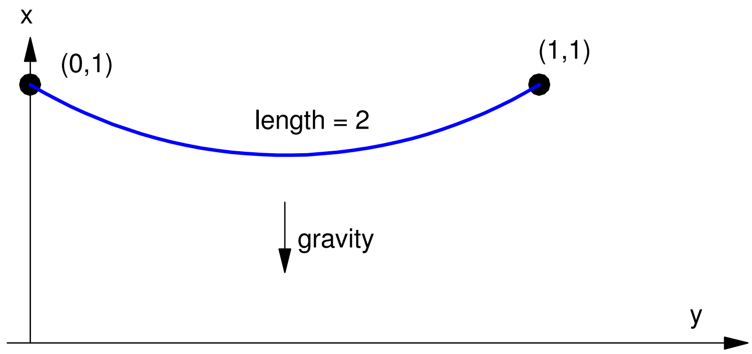
The functional to minize is:

$$F = x\sqrt{1 + y'^{\,2}}$$

y(0) = 1; y(1) = 1

Subject to the constraint that the length of the chain is 2

$$\int_0^1 \sqrt{1 + y'^2}\, dx = 2$$



Hanging Chain: Minimize the potential energy (x) with the constraint that the total langth = 2

Solution: Add a LaGrange multuiplier

$$F = x\sqrt{1+y'^{\,2}} + M\sqrt{1+y'^{\,2}}$$

Plugging into the Euler Legrange equation:

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

There is no y, so

$$F_{y'} = a$$

$$\frac{(x+M)y'}{\sqrt{1+y'^2}} = a$$

Solve for y'

$$(x+M)^2 y'^2 = a^2(1+y'^2)$$

$$\left((x+M)^2 - a^2\right)^2 y'^2 = c^2$$

$$y' = \frac{a}{\sqrt{(x+M)^2 - a^2}}$$

Integrating

$$dy = \frac{a}{\sqrt{(x+M)^2 - a^2}} dx$$

$$\int dy = \int \frac{a}{\sqrt{(x+M)^2 - a^2}} dx$$

Do a change of variables

$$x + M = z$$

$$dx = dz$$

then

$$\int dy = \int \frac{a}{\sqrt{z^2 - a^2}} dx$$

Do another change in variables

$$z = a \cdot \cosh(w)$$

$$dz = a \cdot \sinh(w) \cdot dw$$

Then

$$\int dy = \int \frac{a}{\sqrt{a^2 \cosh^2(w) - a^2}} \cdot a \cdot \sinh(w) \cdot dw$$

$$\int dy = \int \frac{1}{\sqrt{\cosh^2(w) - 1}} \cdot a \cdot \sinh(w) \cdot dw$$

From trig identities:

$$\cosh^2 - 1 = \sinh^2$$

$$\int dy = \int a \cdot dw$$

$$y = aw + b$$

Back substituting

$$y = a \cdot \operatorname{arccosh}\left(\frac{z}{a}\right) + b$$

$$y = a \cdot \text{arccosh}\left(\frac{x+M}{a}\right) + b$$

or

$$\boxed{x = a \cdot \cosh\left(\frac{y-b}{a}\right) - M}$$

Shape of a rope with gravity in the -y direction

With a change of variable, you can also get the shape of a rope with gravity in the -x direction:

$$\boxed{y = a \cdot \cosh\left(\frac{x-b}{a}\right) - M}$$

Shape of a rope with gravity in the -y direction

I get confused if the axis are flipped. Let's rotate the system so y=f(x) and gravity is in the -y direction:

$$y(x) = a\cosh\left(\frac{x-b}{a}\right) - M$$

(x0, y0) = (0, 1)

(1) $$1 = a\cosh\left(\frac{-b}{a}\right) - M$$

(x1, y1) = (1, 1)

(2) $$1 = a\cosh\left(\frac{1-b}{a}\right) - M$$

The third equation comes from the total length being 2 meters:

$$\int \sqrt{1+y'^2} \cdot dx = L$$

$$y = a\cosh\left(\frac{x-b}{a}\right) - M$$

$$y' = -\sinh\left(\frac{x-b}{a}\right)$$

$$\int \sqrt{1+\sinh^2\left(\frac{x-b}{a}\right)} \cdot dx = L$$

$$\int \sqrt{\cosh^2\left(\frac{x-b}{a}\right)} \cdot dx = L$$

$$\int \cosh\left(\frac{x-b}{a}\right) \cdot dx = L$$

$$\left(a\sinh\left(\tfrac{x-b}{a}\right)\right)_{x_0}^{x_1} = L$$ general solution

$$\left(a\sinh\left(\tfrac{x-b}{a}\right)\right)_{0}^{1} = 2$$

(3) $$a\sinh\left(\tfrac{1-b}{a}\right) - a\sinh\left(\tfrac{-b}{a}\right) = L$$ solution here

This gives 3 equations for 3 unknowns. Solving in Matlab

Solving 3 equations for 3 unknowns in MATLAB

First, set up a cost function to return the sum-squared error for (a, b, M)

```
function J = cost3(z)
 a = z(1);
 b = z(2);
 M = z(3);

 % assume gravity is in the -y direction
 %  y = f(x)

 Length = 2;
 x1 = 0;
 y1 = 1;

 x2 = 1;
 y2 = 1;

 e1 = a*cosh((x1-b)/a) - M - y1;
 e2 = a*cosh((x2-b)/a) - M - y2;
 e3 = a*sinh((x2-b)/a)  - a*sinh((x1-b)/a) + - Length;

 x = [x1:0.001:x2]';
 y = a*cosh( (x-b)/a ) - M;
 plot(x,y);
 pause(0.01);

 J = e1^2 + e2^2 + e3^2;

 end
```

Next, use *fminsearch()* to solve

```
>> [A,B] = fminsearch('cost3',10*rand(3,1)-5)

A =

   -0.2296
    0.5000
   -2.0260

B =

  2.1366e-008
```

The cost is close to zero so the answer should be OK. The shape is thus:

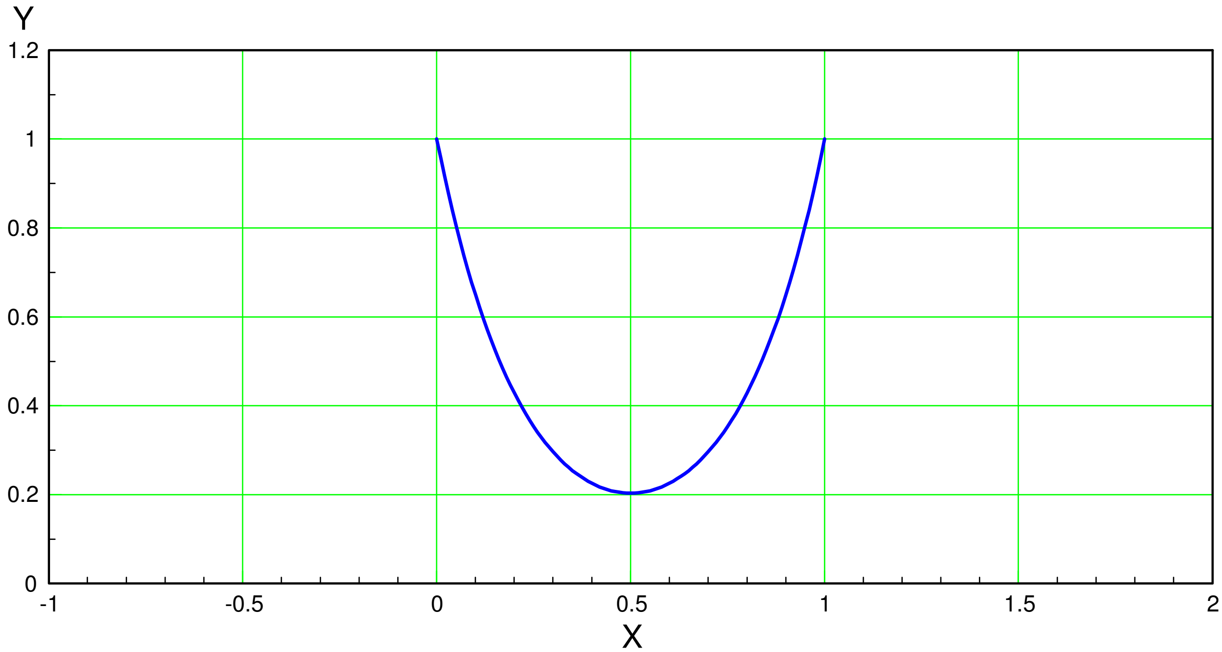
$$\boxed{y(x) = -0.2296\cosh\left(\frac{x-0.5}{-0.2296}\right) - 2.0260}$$

Plotting this:

```
>> a = A(1);
>> b = A(2);
>> M = A(3);
>> x = [0:0.001:1]';
>> y = a*cosh( (x-b)/a ) - M;
>> plot(x,y)
```

This is actually the shape for gravity in the +y direction (a maximum energy). The minimum is the mirror image:

```
>> plot(x,2-y)
```



Shape of a hanging chain of length 2 between (0,1) and (1,1)