Optimal Control and the Ricatti Equation

Calculus of Variations with Dynamic Systems

Recall, a function which minimizes the functional

$$J(x) = \int_a^b F(t, x, \dot{x}) dt$$

must also satisfy the Euler Legrange equation

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

Example: Find x(t) which minimizes the functional

$$J = \int_0^1 (x^2 + \dot{x}^2) dt$$

subject to the constraings that x(0) = 1, x(1) = 0

Solution: The Euler Legrange equation gives

$$F = x^2 + \dot{x}^2$$

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \frac{d}{dt}(2\dot{x}) = 0$$

$$x - \ddot{x} = 0$$

Using LaPlace notation

$$(1 - s^2)x = 0$$

Either x = 0 (the trivial solution) or s = {+1, -1}. The general solution is then

$$x(t) = ae^t + be^{-t}$$

Plugging in the boundary conditions gives

$$x(0) = 1 = a + b$$

$$x(1) = 0 = 2.7183a + 0.3679b$$

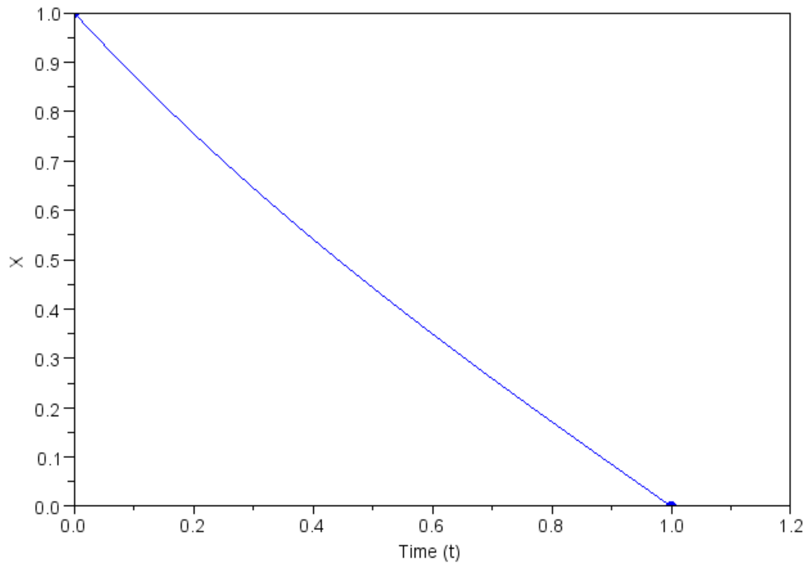
or

a = -0.1565, b = 1.1565

so the funciton which minimizes this funcitonal is

$$x(t) = -0.1565e^t + 1.1565e^{-t}$$

```
-->t = [0:0.001:1]';
-->x = -0.1565*exp(t) + 1.1565*exp(-t);
-->plot(t,x);
-->xlabel('Time (t)');
-->ylabel('X');
```



Optimal path of x(t) with the cost function $J = \int_0^1 \left(x^2 + \dot{x}^2 \right) dt$

Euler Legrange Equation with Two Dependent Variables

If you have *two* dependent variables:

$$J = \int_a^b F(t, x, \dot{x}, u, \dot{u}) dt$$

you have Itwo Euler Legrange equations to solve

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$F_u - \frac{d}{dt}(F_{\dot{u}}) = 0$$

Euler Legrange Equation with Contraints:

Finally, if you have constraints, such as

$$G(t, x, \dot{x}, u, \dot{u}) = 0$$

you can modify the const functional by adding a Legrange multiplier:

$$J = \int_a^b (F(t, x, \dot{x}, u, \dot{u}) + MG(t, x, \dot{x}, u, \dot{u}))dt$$

You can then solve this functional by plugging in the boundary conditions and the constraint on G(t,x,x').

Example 2: Find x(t) to minimize

$$J = \int_0^1 (x^2 + u^2)dt$$

subject to the constraints

$$\dot{x} = u$$

x(0) = 1

x(1) = 0

Solution: Add a Legrange multiplier so that F becomes

$$F = x^2 + u^2 + m(\dot{x} - u)$$

You now have three sets of Euler LaGrange equations to solve:

i) With respect to x:

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \frac{d}{dt}(m) = 2x - \dot{m} = 0$$

ii) With respect to u:

$$F_u - \frac{d}{dt}(F_{\dot{u}}) = 0$$

$$2u - m = 0$$

iii) With respect to m:

$$F_m - \frac{d}{dt}(F_{\dot{m}}) = 0$$

$$\dot{x} - u = 0$$

Solving: From ii)

$$m = 2u$$

$$\dot{m} = 2\dot{u}$$

From iii)

$$u = \dot{x}$$

$$\dot{u} = \ddot{x}$$

Substitute into i)

$$2x = \dot{m} = 2\dot{u} = 2\ddot{x}$$

or

$$\ddot{x} = x$$

or in LaPlace notation

$$\ddot{x} - x = 0$$

$$(s^2 - 1)x = 0$$

This has solutions of

- x = 0 (trivial solution), or
- s = { +1, -1 }

so

$$x(t) = ae^{t} + be^{-t}$$

Plugging in the constraints

$$x(0) = 1 = a + b$$

$$x(1) = 0 = 2.7183a + 0.3679b$$

results in

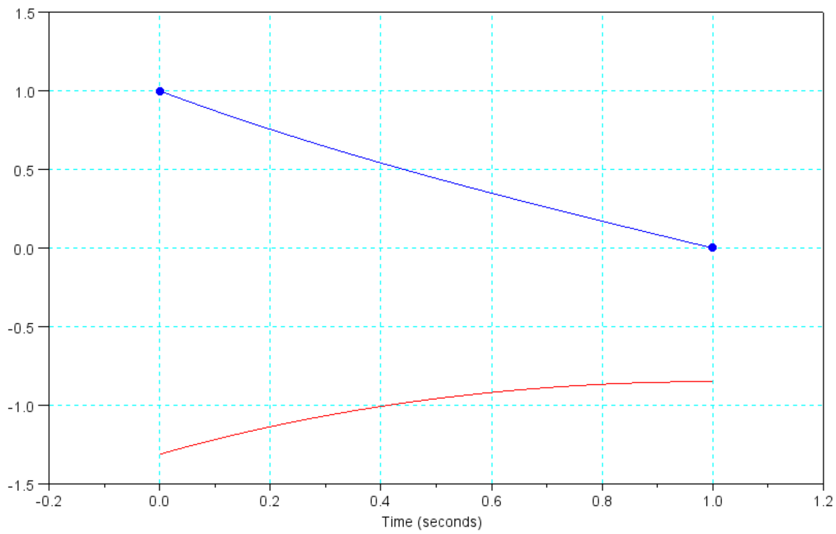
a = -0.1565

b = 1.1565

and

$$x(t) = -0.1565e^{t} + 1.1565e^{-t}$$

$$u(t) = \dot{x}(t) = -0.1565e^{t} - 1.1565e^{-t}$$



Optimal path for x(t) (blue) and input u(t) (red) for cost function $J = \int_0^1 \left(x^2 + u^2\right) dt$

Note: If you change the funcitonal to weight x more heavily, it is driven to zero quicker:

$$J = \int_0^1 (100x^2 + u^2) dt$$

The functional becomes:

$$F = 100x^2 + u^2 + m(\dot{x} - u)$$

which results in the following three Euler LaGrange equations:

$$200x - \dot{m} = 0 \qquad \text{(partials with respect to x)}$$

$$2u - m = 0 \qquad \text{(partials with respect to u)}$$

$$\dot{x} - u = 0 \qquad \text{(partials with respect to m)}$$

which simplifies to:

$$\ddot{x} - 100x = 0$$

or

$$(s^2 - 100)x = 0$$

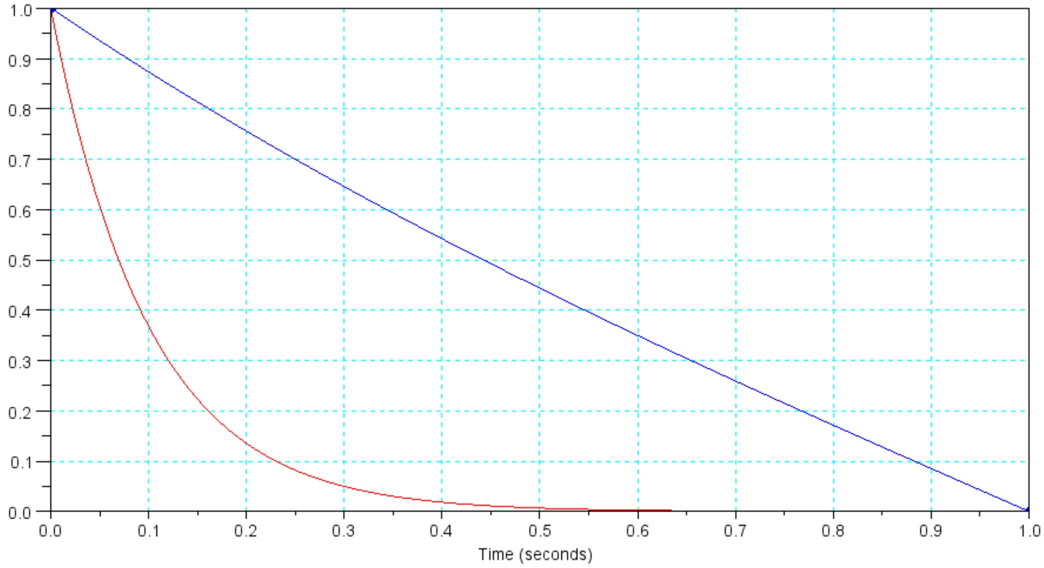
$$s = \pm 10$$

meaning

$$x(t) = ae^{10t} + be^{-10t}$$

which, after solving for the initial conditions, becomes:

$$x = 0.000000002e^{10t} + 1e^{-10t}$$



Optimal Path for $J = \int_0^1 \left(x^2 + u^2\right) dt$ (blue) and $J = \int_0^1 \left(100x^2 + u^2\right) dt$ (red)

Example 3: Find the functional to minimize

$$J = \int_a^b (X^T Q X + U^T R U) dt$$

subject to the constraint

$$\dot{X} = AX + BU$$

Solution: The functional becomes with a LaGrange multiplier:

$$F = (X^T Q X + U^T R U) + 2M^T \left(AX + BU - \dot{X}\right)$$

The Euler Legrange equations are then

$$2X^T Q + 2M^T A - \frac{d}{dt}(-2M^T) = 0$$

$$\dot{M}^T = X^T Q - M^T A$$

$$\dot{M} = -QZ - A^T M$$

and

$$2U^T R + 2M^T B = 0$$

$$RU = -B^T M$$

$$U = -R^{-1}B^T M$$

so you have the dyamic system

$$\begin{bmatrix} \dot{X} \\ \dot{M} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ M \end{bmatrix}$$

which can be solved subject to the constraints on X(a) and X(b)

Full-State Feedback Formulation:

Assume that

$$M = PX$$

so that the full-state feedback gains are

$$K = R^{-1}B^T P$$

Then the dynamics become

$$\dot{X} = (A - BR^{-1}B^T P)X$$

$$\dot{P}X + P\dot{X} = (-Q - A^T P)X$$

$$P\dot{X} = \left(-\dot{P} - Q - A^T P\right)X$$

This implies that

$$PA - PBR^{-1}B^T P = -\dot{P} - Q - A^T P$$

or

$$\boxed{\begin{aligned} \dot{P} &= -A^T P - PA - Q + PBR^{-1}B^T P \\ K &= -R^{-1}B^T P \end{aligned}}$$

Algebraic Ricatti equation for computing the time-varying feedback gains: U =-KX

This gives the optimal time-varying feedback gain. If the feedback gains are to be constants, then

$$\dot{P} = 0$$

and

$$\boxed{\begin{aligned} 0 &= -A^T P - PA - Q + PBR^{-1}B^T P \\ K &= -R^{-1}B^T P \end{aligned}}$$

Algebraic Ricatti equation you'll see in most places

Example: For the first-order system

$$\dot{x} = u$$

$$J = \int_0^\infty (qx^2 + ru^2)dt$$

m is

$$0 = -m^2 / r + q$$

or

$$m = \sqrt{qr}$$

$$k = \sqrt{q / r}$$

Note that

- Only the ratio of q/r matters - not their absolute values. This is reasonable since U(t) minimizes a functional. The minimum of F() will also be the minimum of 10F().
- As Q increases, the poles shift left (faster) as the square root of Q
- As R incrases, the poles shift right (slower) as the square root of R

Example 4:

$$\dot{x} = -x + u$$

$$J = \int_0^\infty (x^2 + u^2)dt$$

$$q = 1$$

$$r = 1$$

The Ricatti equation becomes

$$0 = -A^T P - PA - Q + PBR^{-1}B^T P$$

$$0 = -2p - 1 + p^2$$

$$p = \{\ 0.4142,\ \ -2.4142\ \}$$

$$k = \{\ 0.4142\ \ -2.4142\ \}$$

This is a typical result.

- P (the Ricatti equation) is a quadratic equation - hence generally there are two solutions
- One of these solutions will be a minimum, the other a maximum. Since the feedback gain of -2.4142 results in an unstable system, that is the wrong solution (the maximum). Select the one that stabilizes the system.

The optimal feedback gain is

$$k = 0.4142$$

$$u = -kx$$