LQG Control

With pole placement techniques, you can place the closed-loop system's poles where you like. But, where *should* the poles be placed? Alternatively, what is the "best" set of feedback gains?

To answer that you have to define a measure of "best."

- If all you care about is minimum time, the result will be an arbitrarily large input.
- If all you care about is minimizing the control effort (read fuel, power, etc), you tend to get U = 0.
- If you want to minimize both the settling time as well as the control input, you wind up with a trade-off.

Linear Quadratic Gaussian Control (LQG) is a way to do the latter.

Control Solution:

Assume you have a linear system with an arbitrary initial condition

$$sX = AX + BU$$

$$Y = CX$$

Define a cost function

$$J = \int_0^\infty (X^T Q X + U^T R U) dt$$

The optimal solution will be

$$U = -K_x X$$

where

$$K_x = R^{-1} B^T P$$

and P is the solution to the Ricatti equation

$$\dot{P} = -A^T P - PA - Q + PBR^{-1} B^T P$$

or in steady-state:

$$0 = -A^T P - PA - Q + PBR^{-1} B^T P$$

Comments:

- Essentially, the cost function is the matrix form of

$$J = \int_0^\infty \left(\sum q_i x_i^2 + \sum r_i u_i^2 \right)$$

- In order for the cost to be finite, both the states (X) and input (U) must go to zero as $t \to \infty$. This assures that the closed-loop system is stable (assuming it is possible to stabilize it).
- The nice thing about choosing this cost function is there is a closed-form solution (a big plus) with constant gains (also good) which is full-state feedback.
- You could use other cost functions - but that would make the solution *much* harder to obtain.

Sometimes, this solution is termed *optimal control*. I personally don't like that name since it is optimal only for that particular cost function. Plus, it has been shown that *any* stabilizing control law is optimal for some Q and R. Hence, all stabilizing control laws are optimal. That sort of makes the word *optimal* meaningless.

I prefer to think this as another way to find feedback gains - similar to pole placement.

- The disadvantage is you lose control of where the resulting poles are placed.
- The advantage is you tend to get lower gains and better behaving systems than pole placement.

SciLab Code

```
P = ricc(A,B*inv(R)*B',Q,'cont');
Kx = inv(R)*B'*P
```

MATLAB Code

```
Kx = lqr(A, B, Q, R)
```

Example: Heat Equation

Design a feedback controller for the 4th-order heat equation to optimize the cost function

a) $J = \int_0^\infty (y^2 + u^2) dt$

b) $J = \int_0^\infty (10^4 y^2 + u^2) dt$

c) $J = \int_0^\infty (y^2 + 10^4 u^2) dt$

Solution:

First, input the system (A, B)

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

  - 2.    1.    0.    0.
    1.  - 2.    1.    0.
    0.    1.  - 2.    1.
    0.    0.    1.  - 1.

B = [1;0;0;0]

    1.
    0.
    0.
    0.
```

Define the weighting matrices (Q, R)

```
C = [0,0,0,1]

    0.    0.    0.    1.
```

```
Q = C'*C

     0.     0.     0.     0.
     0.     0.     0.     0.
     0.     0.     0.     0.
     0.     0.     0.     1.

R = 1;
```

Solve the Ricatti equation to find the full-state feedback gains:

```
Kx = lqr(A, B, Q, R)

     0.0426938     0.0862990     0.1279791     0.1572416
```

So, with this weighting (Q, R), the "optimal" location of the closed-loop poles are:

```
eig(A-B*Kx)

  -3.5322276
  -2.3461035
  -1.0089118
  -0.0700632
```

Repeating for weights of $\{10^4, 10^2, 1, 10^{-2}, 10^{-4}\}$ for Q:

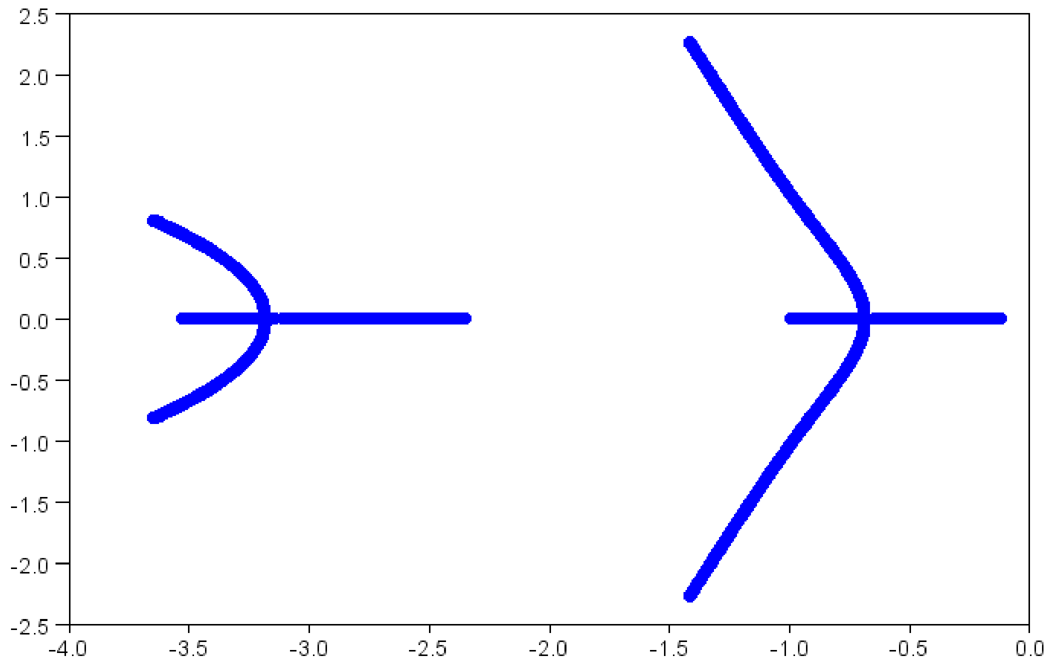
Q	"Optimal" Feedback Gain: Kx
10^{-4}	0.0000053 0.0000105 0.0000154 0.0000188
10^{-2}	0.0005257 0.0010516 0.0015389 0.0018713
1	0.0426938 0.0862990 0.1279791 0.1572416
10^{2}	0.6842631 1.6026342 2.8230412 3.9399371
10^{4}	3.128746 11.152018 29.584146 55.140089

Note that as the output becomes more and more important (or, relatively, the input becomes less and less important), the gains increase

The corresponding location of the closed-loop poles are:

Q	"Optimal" Location of Closed-Loop Poles
10^{-4}	-3.53, -2.34, -0.99, -0.12
10^{-2}	-3.53, -2.34, -0.99, -0.12
1	-3.53 -2.34, -0.99, -0.17
10^{2}	-3.51, -2.45, -0.85 + j0.65
10^{4}	-3.64 + j0.81, -1.41 + j2.27

Not that it means much, but the 'optimal' location of the closed-loop poles following the following plot (repeating for 1000 weightings between 10^{-4} and 10^{4})

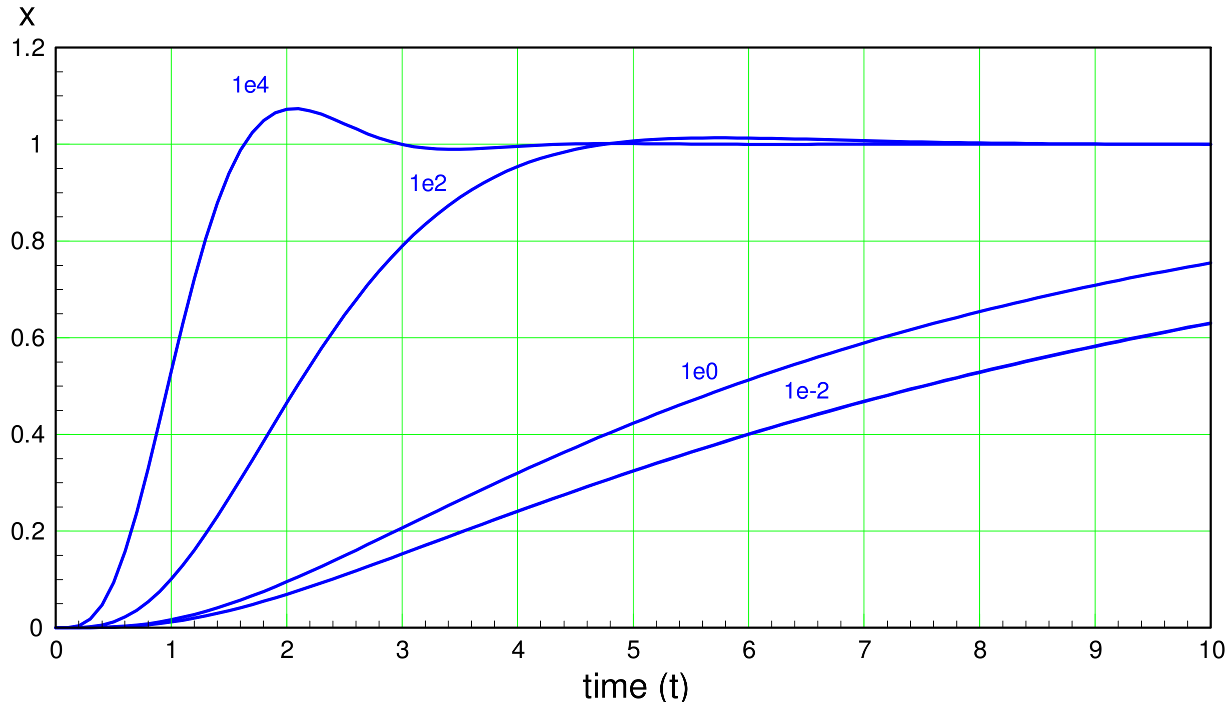


Location of "Optimal" Closed Loop Poles for R=1, $10^{-4} < Q < 10^{4}$

The step responses comes from the following Matlab code (with the DC gain set to one)

```
Kx = lqr(A, B, Q, R);

DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
G = ss(A-B*Kx,B*Kr,C,0);
y = step(G,t);
plot(t,y);
```



"Optimal" Step Response for the Weighting on Q Varying from 10^{-4} to 10^{+4}

Note that as the output becomes more and more important, it is driven to its final value faster and faster. Also note that zero is an arbitrary number. If you change the initial conditions so that the system starts at -1 and is driven to zero, the control law minimizes the cost function

$$J = \int_0^\infty (q \cdot y^2 + r \cdot u^2) dt$$

Tuning the Step Response:

To tune the step response, Q and R can be adjusted. Note that the output is

$$Y = CX$$

To weight the output, let

$$Q = C^T C$$

If you want to penalize velocity (sort of like adding friction),

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAX$$

$$Q = (CA)^T CA$$

Likewise, let R=1 and adjust Q to get the response you desire:

- If the system is too slow, increase Q = CTC
- If the system is too fast, decrease Q = CTC

- If you want to add friction, add a term:

$$Q = \alpha C^T C + \beta (CA)^T CA$$

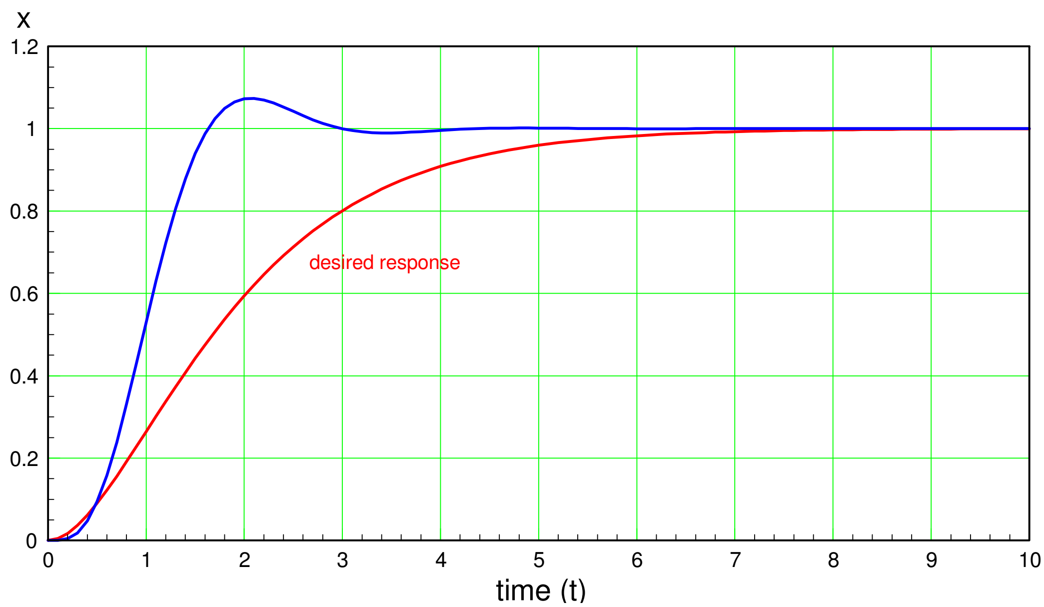
Example: Design a feedback controller so that the 4th-order heat equation has

- No overshoot for a step input, and
- A 2% settling time of 4 seconds

Solution:

Step 1: Adjust $Q = C^T C$ until you get the speed you want:

$$Q = 10^4 \ C^T C$$

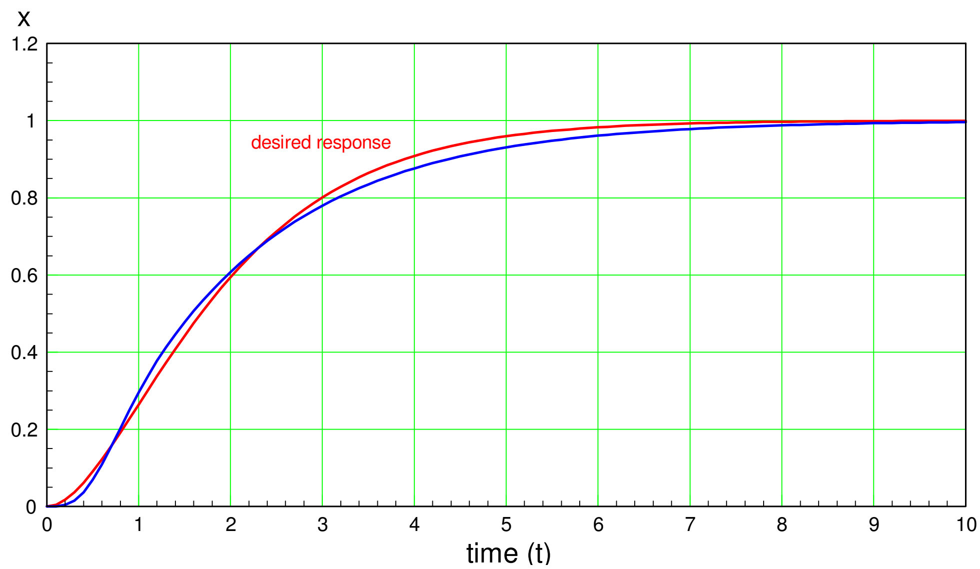


Step response for $Q = 10^4 C^T C$ (blue) as well as the desired response

Step 2: Add friction to get rid of the overshoot

$$Q = 10^4 \cdot C^T C + 10^4 (CA)^T CA$$

where the second term was adjusted until the step response was close to what was desired



Optimal Gain for $Q = 10^4\, C^T C + 3*10^4\, (CA)^T CA$ (blue) and Desired Response (red)