LaPlace Transforms and Dominant Poles

NDSU ECE 463/663

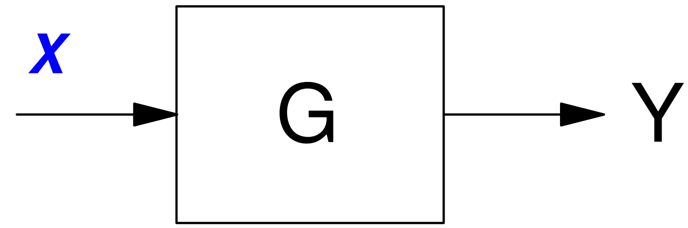
Lecture #2

Inst: Jake Glower

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Static Systems:

$$Y = k * X$$



Static System:

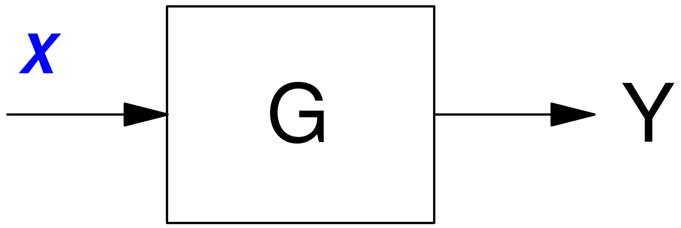
- Y looks just like X, only scaled

This means that:

- There is no memory:
  - The previous values of Y and X do not matter
- A sine wave input produces a sine wave output
- A square wave input produces a square wave output
- A random input produces the identical random output, only changed in amplitude.

Dynamic Systems

$$Y = G(s)\ X$$



Dynamic systems are described by differential equations.

- Dynamics systems have memory:
  - y(0) is needed to find y(t)
- Dynamic systems change the shape of the input
  - The gain changes with frequency
  - An input that is not a sine wave is distorted.

Differential Equations and Transfer Functions

Assume x(t) and y(t) are related by a differential equation:

$$y'' + ay' + by = cx'' + dx' + ex$$

Assume all functions are in the form of

$$y = e^{st}$$

then differentiation becomes multiplication by 's'

$$\frac{dy}{dt} = s \cdot e^{st} = sy.$$

With this assumption,

$$s^2 Y + asY + bY = cs^2 X + dsX + eX$$

$$Y = \left(\frac{cs^2+ds+e}{s^2+as+b}\right) X = G(s) \cdot X$$

Example: Find the transfer function relating X and Y

$$y''' + 3y'' + 2y' + 10y = 20x' + 5x$$

$$(s^3 + 3s^2 + 2s + 10)Y = (20s + 5)X$$

$$Y = \left(\frac{(20s+5)}{\left(s^3+3s^2+2s+10\right)}\right)X$$

$$G(s) = \left(\frac{(20s+5)}{\left(s^3+3s^2+2s+10\right)}\right)$$

Steady-State Solution for Sinusoidal Inputs (phasors):

- If x(t) is a sinusoid, phasors can be used to find y(t)

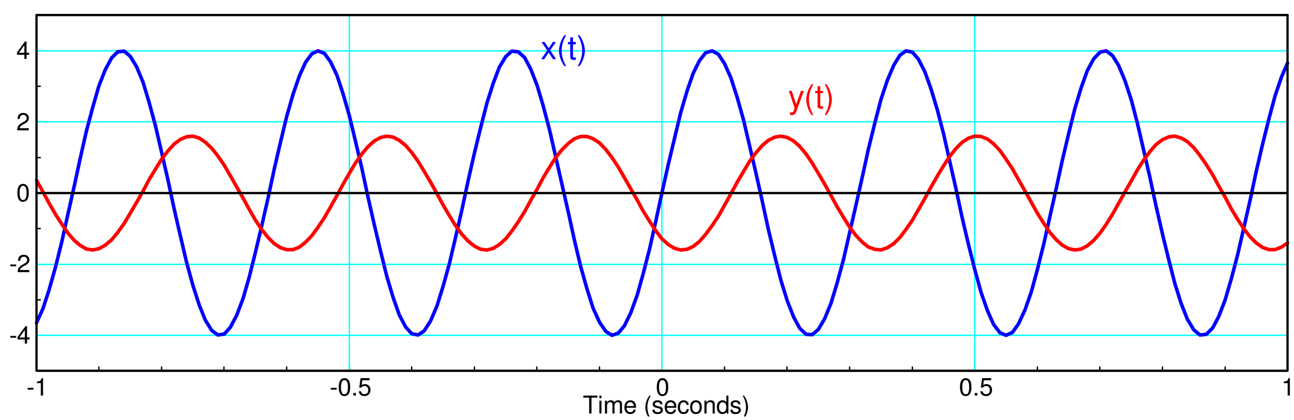
Example: Find y(t) assuming

$$Y = \left(\frac{200}{s^2+20s+100}\right)X \qquad x(t) = 4\sin(20t)$$

Solution: s = j20 $\qquad X = 0 - j4$

$$Y = \left(\frac{200}{s^2+20s+100}\right)_{s=j20}(0-j4) = -1.280 + j0.960$$

$$y(t) = -1.280\cos(20t) - 0.960\sin(20t)$$



Transient Solutions: LaPlace Transforms

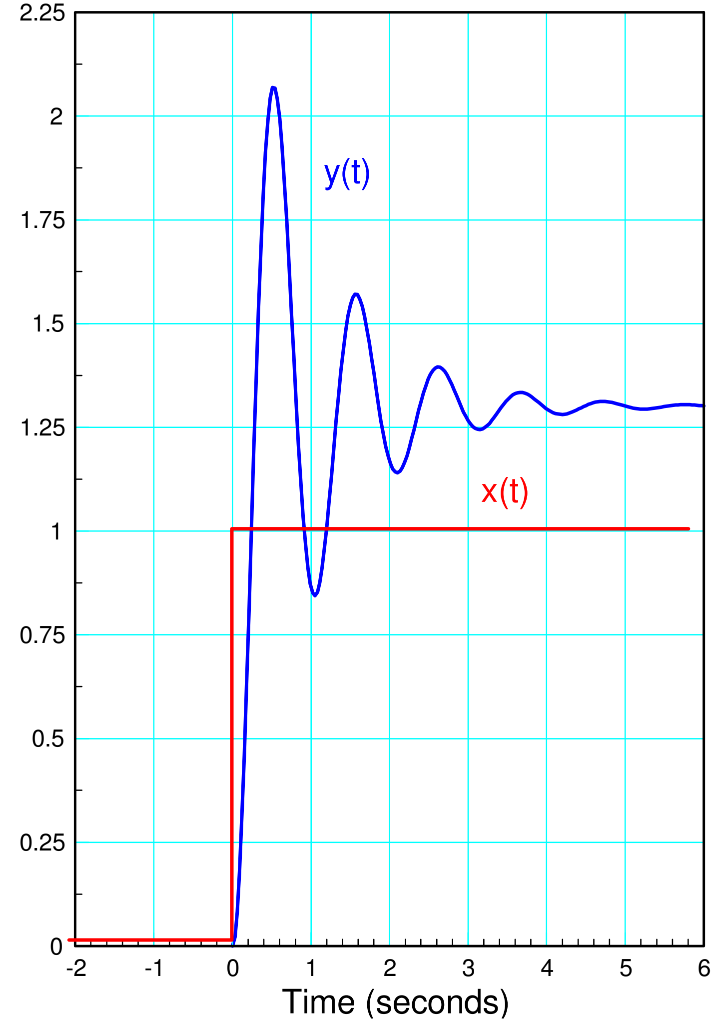
If the input is zero for t<0

$$x = x(t) \cdot u(t)$$

where u(t) is the unit step function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

LaPlace transforms are used to find y(t).



Signals and Systems vs. Modern Control

In ECE 343 Signals, you looked at two-sided two-dimensional LaPlace transforms.

- For image processing, you can go left and right (two-sided)
- For image processing, you can also go up / down (two-dimenstional)

In ECE 463, t represents time

- Time is one-dimensional and
- Time always goes forward (single sided).

Hence, in this class, we look at the mundane case of single-sided one-dimensional LaPlace transforms.

Table of LaPlace Transforms

Only four LaPlace transforms are needed for this class

- With partial fraction expansion, you can then solve any system.

Table 1: Common LaPlace Transforms		
Name	Time: y(t)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	u(t)	$\frac{1}{s}$
exponential	$a \cdot e^{-bt}u(t)$	$\frac{a}{s+b}$
damped sinusoid	$2a \cdot e^{-bt}\cos(ct-\theta)u(t)$	$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right)$

LaPlace Transform Example:

Find y(t):

y'' + 3'y + 2y = 4x   x(t) = u(t)

Solution: Convert to LaPlace notation

$$(s^2 + 3s + 2)Y = 4X$$

$$Y = \left(\frac{4}{s^2+3s+2}\right)X$$

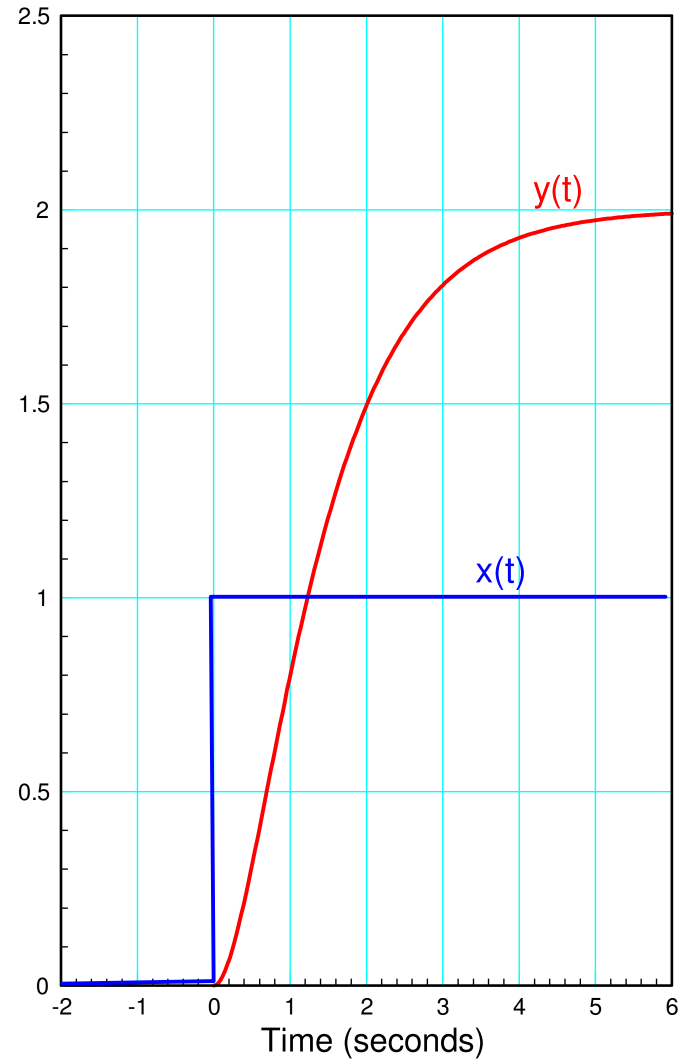
Substitute the LaPlace transform for U(s)

$$Y = \left(\frac{4}{s^2+3s+2}\right)\left(\frac{1}{s}\right)$$

Factor and use partial fractions to expand Y(s)

$$Y = \left(\frac{4}{s(s+1)(s+2)}\right)$$

$$Y = \left(\frac{2}{s}\right) + \left(\frac{-4}{s+1}\right) + \left(\frac{2}{s+2}\right)$$



Use the above table to convert back to y(t)

$$\boxed{y(t) = (2 - 4e^{-t} + 2e^{-2t})u(t)}$$

Example 2: Find the y(t) given that

$$Y(s) = G \cdot X = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution: Factoring Y(s)

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle{-161.56^0}}{s+1+j3}\right) + \left(\frac{0.7906\angle 161.56^0}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos\left(3t + 161.56^0\right) \qquad \text{for } t>0$$

Dominant Poles

Poles = Energy

- If there are N ways to store energy, you have N poles

If there are thousands of ways to store energy, there are thousands of poles

- Dealing with thousandth order systems is cumbersome.

Fortunately, most systems have a few poles which dominate the response

A model which includes the dominant poles will be

- Fairly accurate (good), and
- Low order (also good)

Dominant Pole Example:

Find the step response of

$$Y = \left(\frac{20}{(s+1)(s+10)}\right)X$$

Solution:

$$Y = \left(\frac{20}{(s+1)(s+10)}\right)\left(\frac{1}{s}\right) = \left(\frac{2}{s}\right) + \left(\frac{-2.222}{s+1}\right) + \left(\frac{0.222}{s+10}\right)$$

$$y(t) = 2 - 2.222e^{-t} + 0.222e^{-10t} \qquad t > 0$$

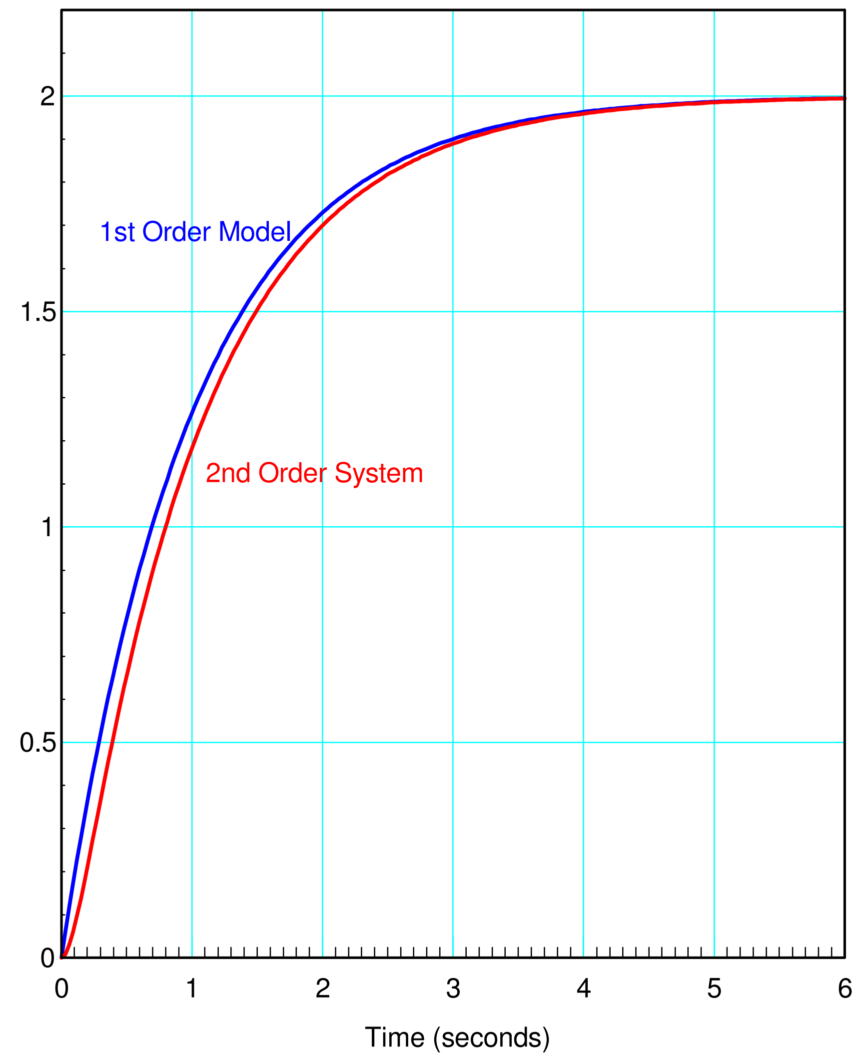
Here, the pole at -1 is dominantes the pole at -10 for two reasons:

- Its initial condition is 10x larger than the pole at s = -10, and
- Its transient response lasts 10x longer than the pole at s = -10

1st-Order Approximation

- Keep the dominant pole (s = -1)
- Match the DC gain

$$\left(\frac{20}{(s+1)(s+10)}\right) \approx \left(\frac{2}{s+1}\right)$$



First-Order approximation

Single (real) dominant pole

$$Y = \left(\frac{a}{s+b}\right)X$$

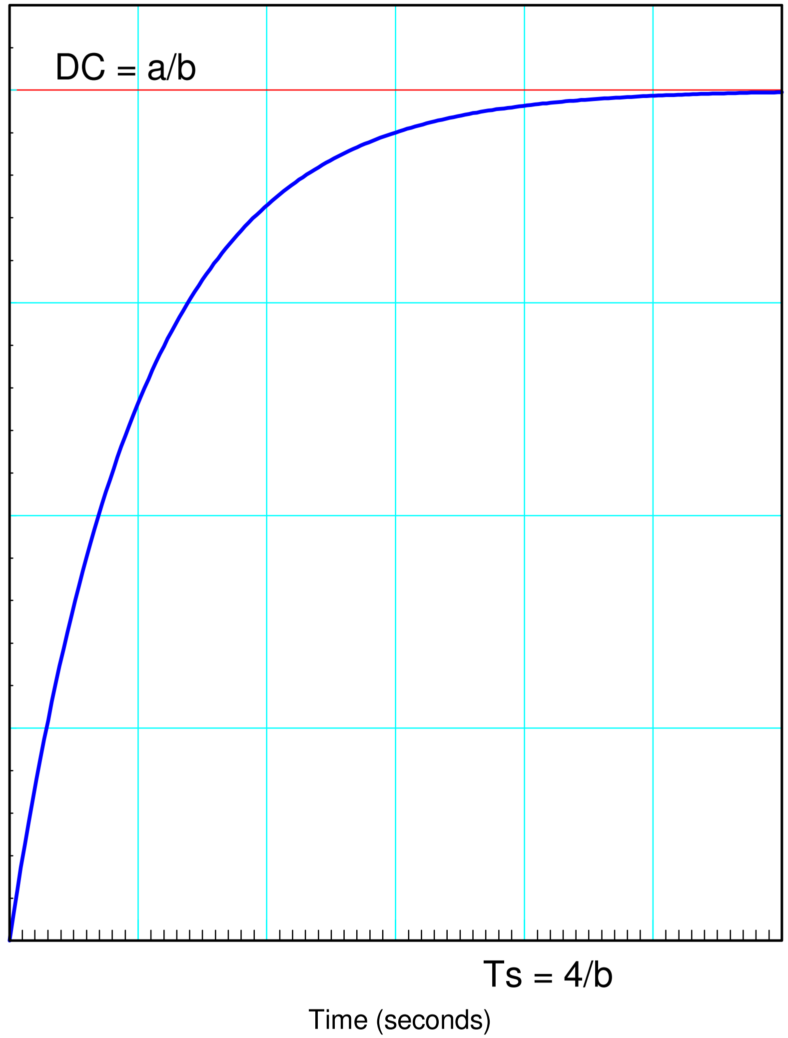
DC Gain:

- $\left(\frac{a}{s+b}\right)_{s=0} = \frac{a}{b}$

2% Settling Time:

$$0.02 = e^{-bt}$$

$$\boxed{T_s = \frac{4}{b}}$$



Example: Sketch the step response

$$Y = \left(\frac{50{,}000}{(s+3)(s+10)(s+20)(s+50)}\right) X$$

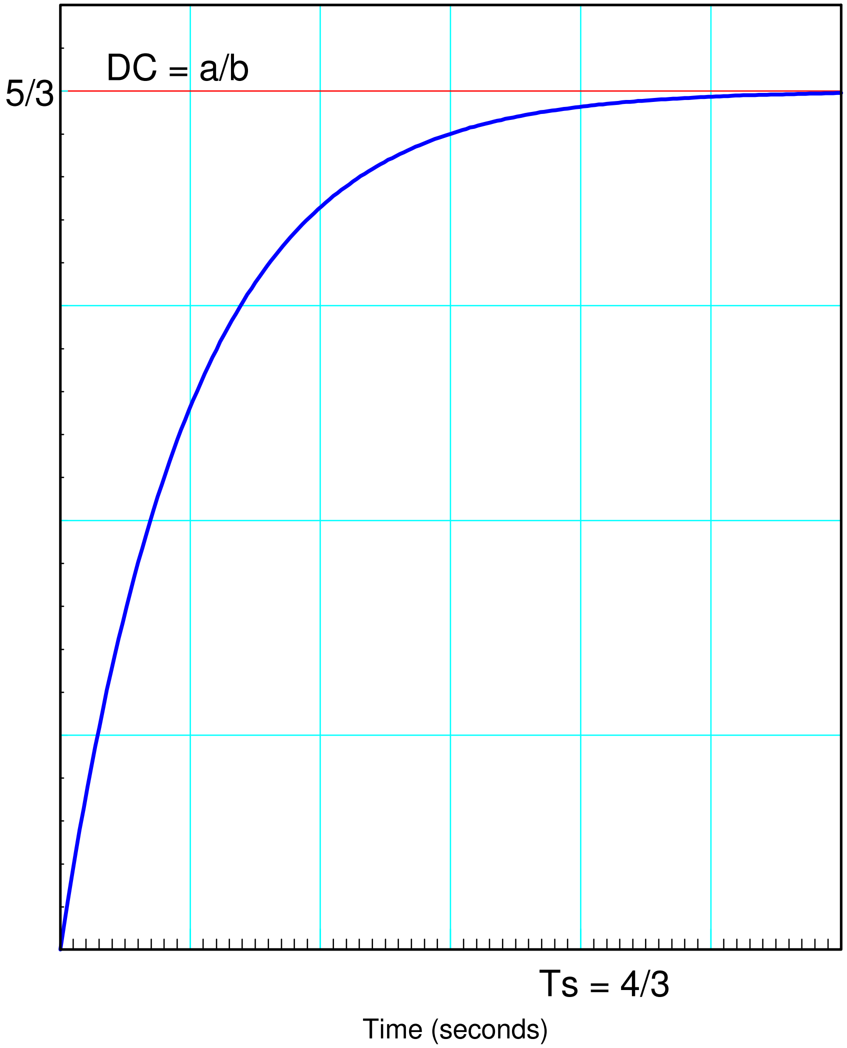
Solution:

- The DC gain is 1.67

$$\left(\frac{50{,}000}{(s+3)(s+10)(s+20)(s+50)}\right)_{s=0} = 1.67$$

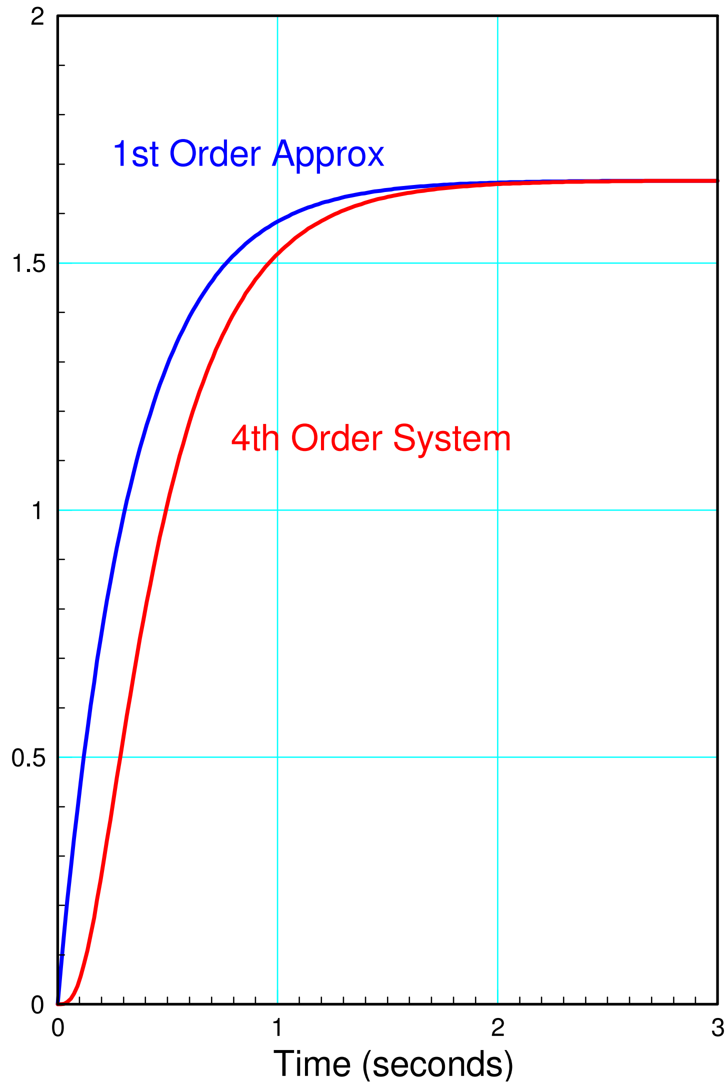
- The dominant pole is s = -3
- Ts = 4/3 sec

$$\left(\frac{50{,}000}{(s+3)(s+10)(s+20)(s+50)}\right) \approx \left(\frac{5}{s+3}\right)$$



Checking with Matlab

```
G4 =
zpk([],[-3,-10,-20,-50],50000)

             50000
--------------------------
(s+3) (s+10) (s+20) (s+50)


G1 = zpk([],[-3],5)

  5
-----
(s+3)

t = [0:0.01:3]';
y1 = step(G1,t);
y4 = step(G4,t);
plot(t,y1,t,y4);
```



Example 2: Find G(s)

1st-Order Approximation

- No oscillations

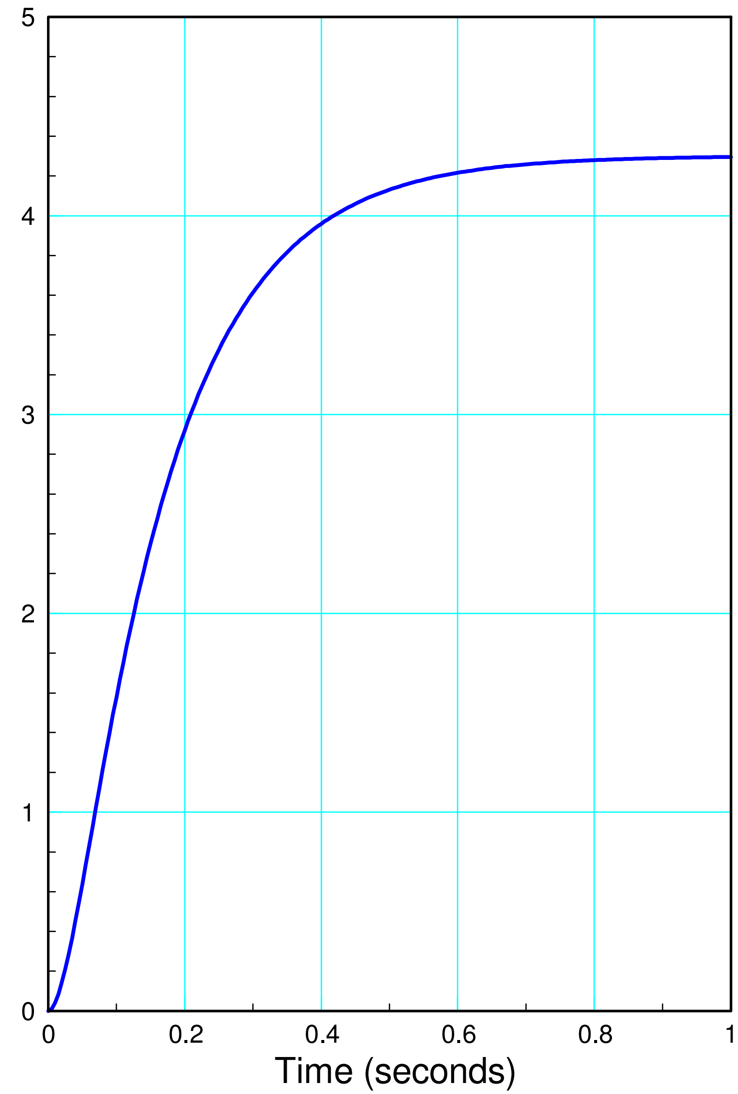
DC gain = 4.3

Ts = 0.57 seconds

$$b = \frac{4}{0.57} = 7.0$$

Putting it together:

$$\boxed{G(s) \approx \left(\frac{30.1}{s+7}\right)}$$



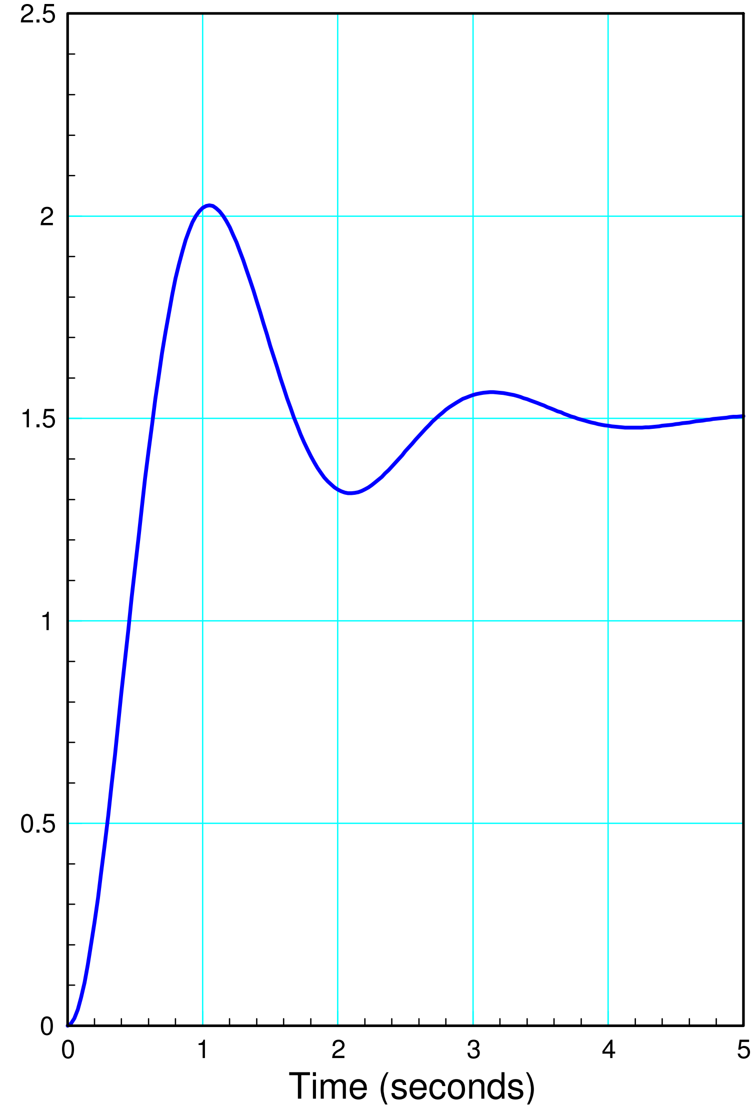
2nd-Order Approximations

Dominant pole is complex

- Plus it's complex conjugate

$$G(s) \approx \left(\frac{k \cdot \omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}\right)$$

$$G(s) \approx \left(\frac{k \cdot \left(\sigma^2 + \omega_d^2\right)}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)}\right)$$



Now we need 3 parameters (pick 3)

DC Gain

- $G(s = 0)$

Frequency of Oscillation

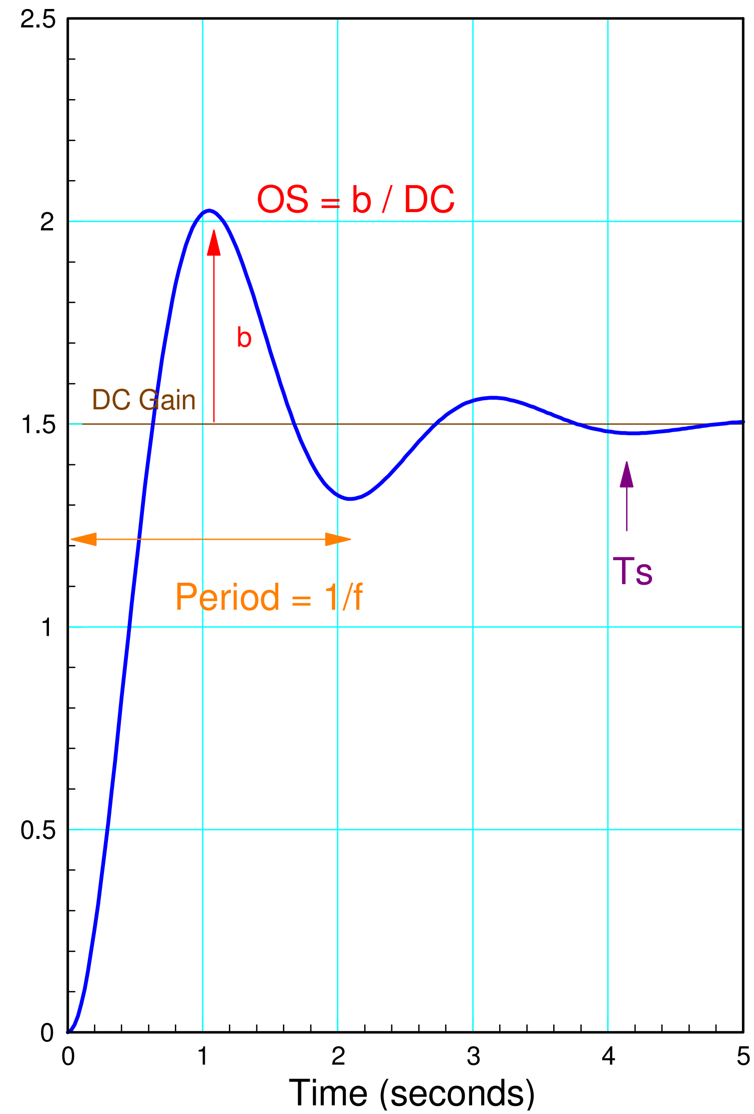
- The complex part of the dominant pole
- $\omega_d = 2\pi f = \frac{2\pi}{T}$

Ts: 2% Settling Time

- The real part of the dominant pole
- $T_s = \frac{4}{\sigma}$

Percent Overshoot

- The angle of the dominant pole
- $OS = \frac{b}{DC} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$
- ζ is the *damping ratio*



Example: Determine the step response of

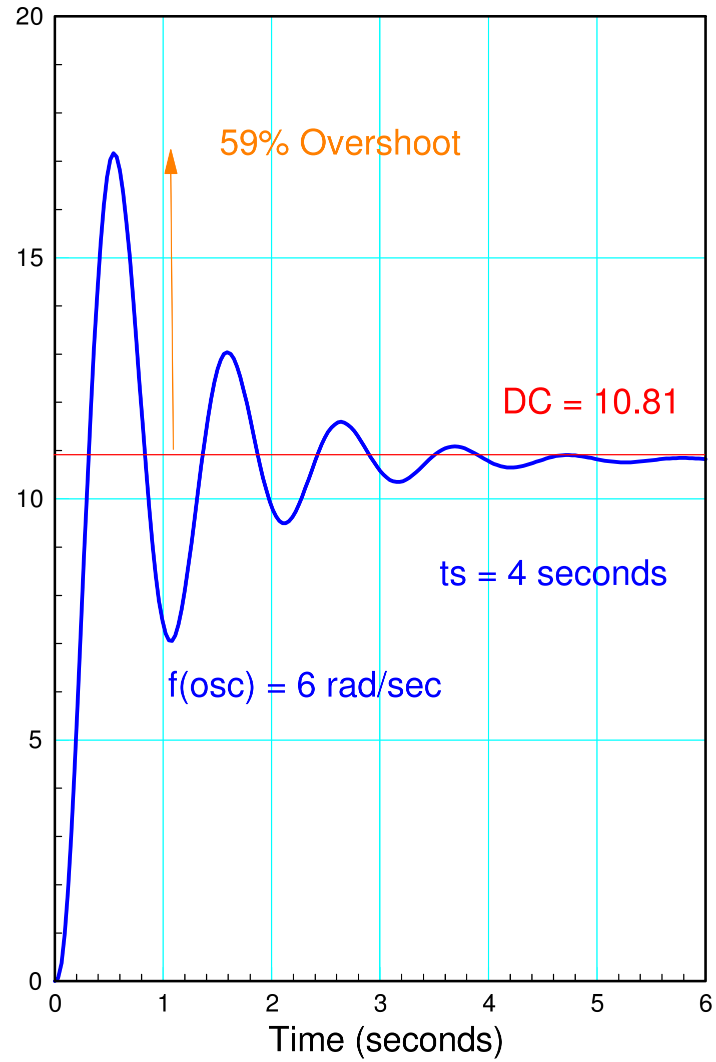
$$Y = \left(\frac{20{,}000}{(s+1+j6)(s+1-j6)(s+50)} \right) X$$

Solution:

- The dominant poles are $s = -1 \pm j6$
- The DC gain is 10.81

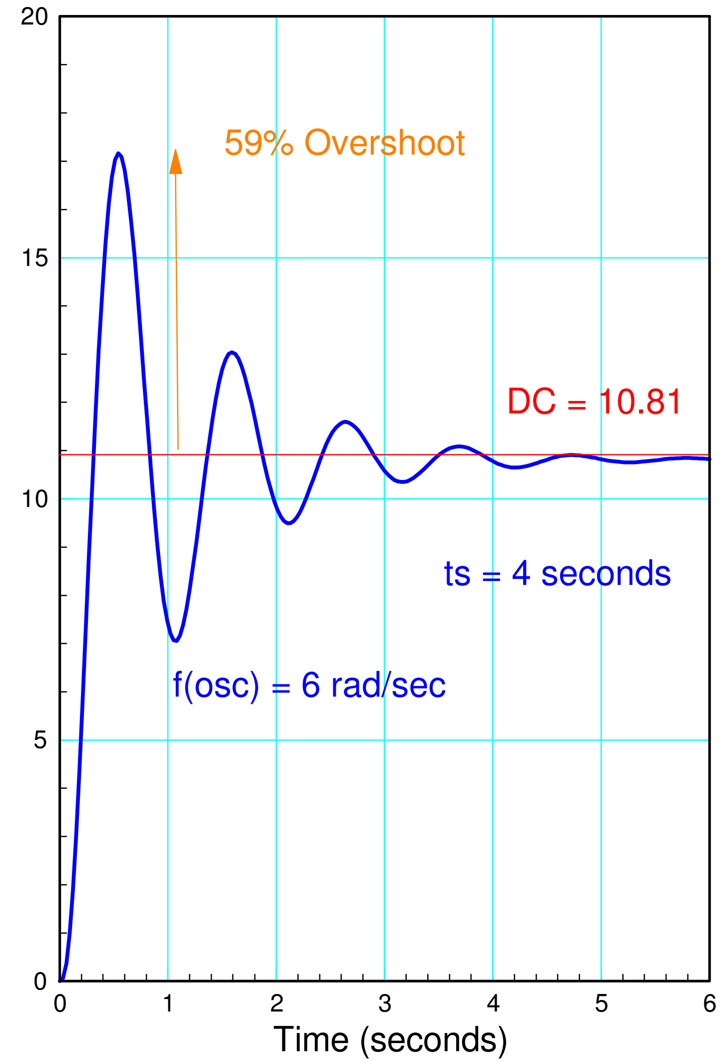
meaning

- y(t) goes to 10.81 (the DC gain)
- Ts = 4 seconds (4/1)
- f(osc) = 6 rad/sec (about 1 Hz)
- ζ = 0.164, meaning
- There will be 59% overshoot



Checking in Matlab

```
G3 =
zpk([],[-1+j*6,-1-j*6,-50],20000)

         20000
----------------------
(s+50) (s^2 + 2s + 37)

t = [0:0.001:10]';
y3 = step(G3,t);
DC = y3(10000)

DC =   10.8113

OS = max(y3) / DC

OS =    1.5879
```



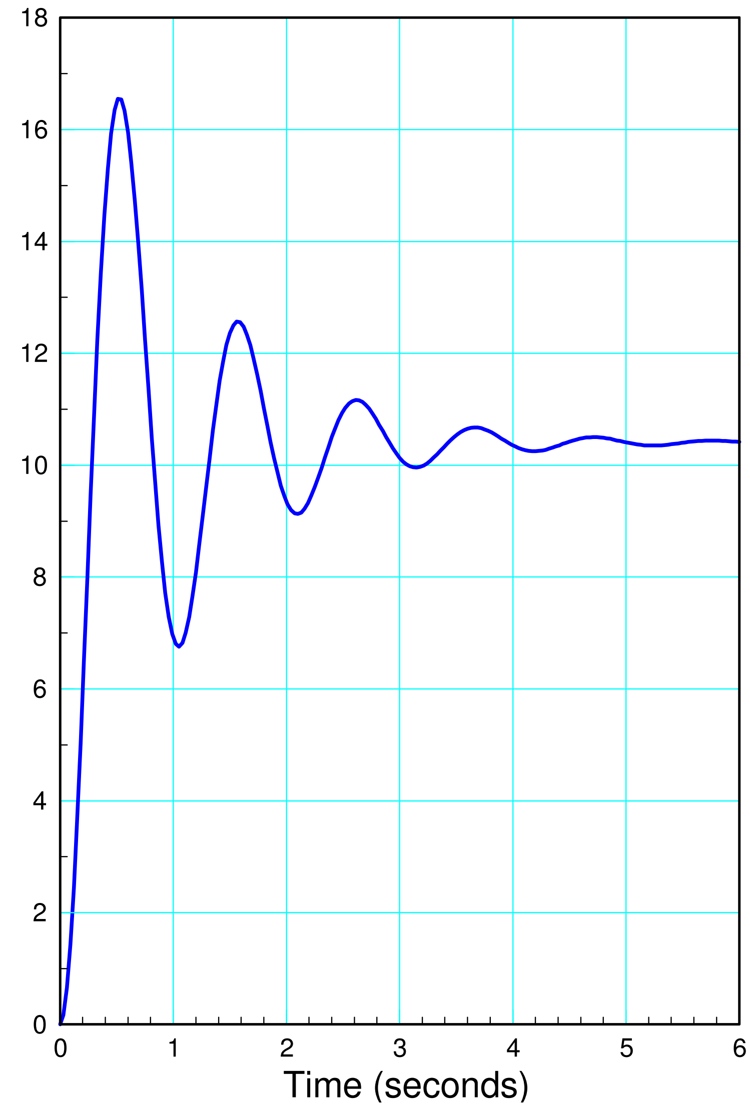
Example 4: The step response of G(s) is shown to the right. Find G(s);

Solution:

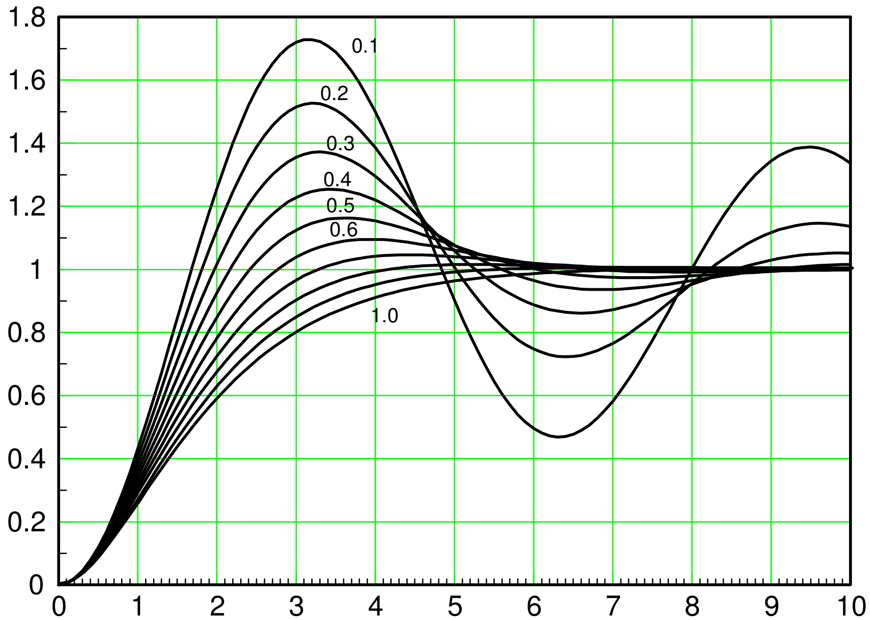
- Second Order (oscillates)
- DC gain = 10.2
- $\omega_d = \left(\frac{3 \text{ cycles}}{3.1 \text{ sec}}\right) 2\pi$
- $T_s = 5 \text{ sec } = \frac{4}{\sigma}$
- $OS = \left(\frac{16.6-10.2}{10.2}\right) = 0.627$

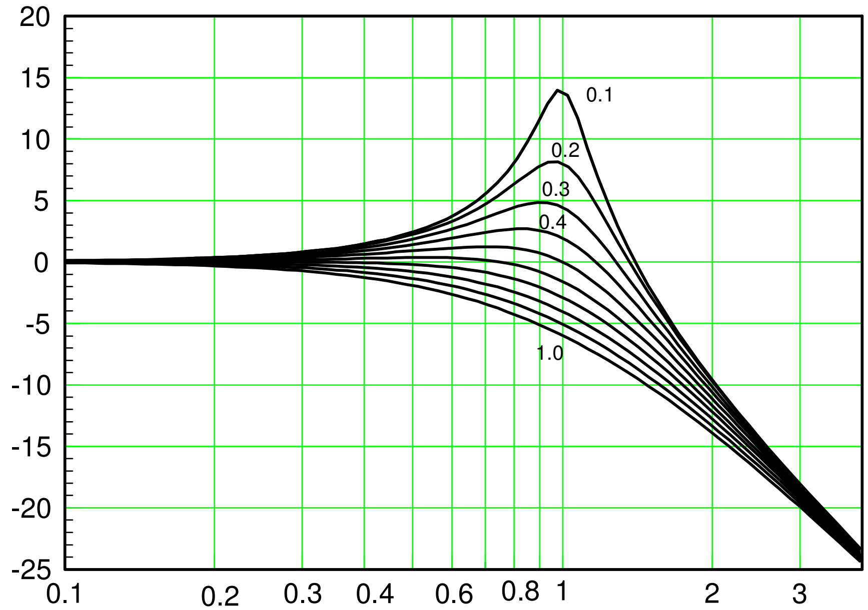
Pick 3 to find G(s)

$$G(s) \approx \left(\frac{386}{(s+0.8+j6.1)(s+0.8-j6.1)}\right)$$



Second Order Approximations





zeta	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
%OS	100%	73%	53%	37%	25%	16%	9%	5%	1.5%	0.1%	0%
Mm	inf	5.02	2.55	1.75	1.36	1.15	1.04	1	1	1	1