System Modeling and State-Space

NDSU ECE 463/663

Lecture #3

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Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

System Modeling

Transfer Function: $Y = \left(\frac{10}{s^2+2s+10}\right)X$

- Used in Circuits, Electronics, Signals and Systems, Controls System
- Leads to Classic Control methods (root locus, Bode plots, etc.)

State Space

$$sX = AX + BU$$

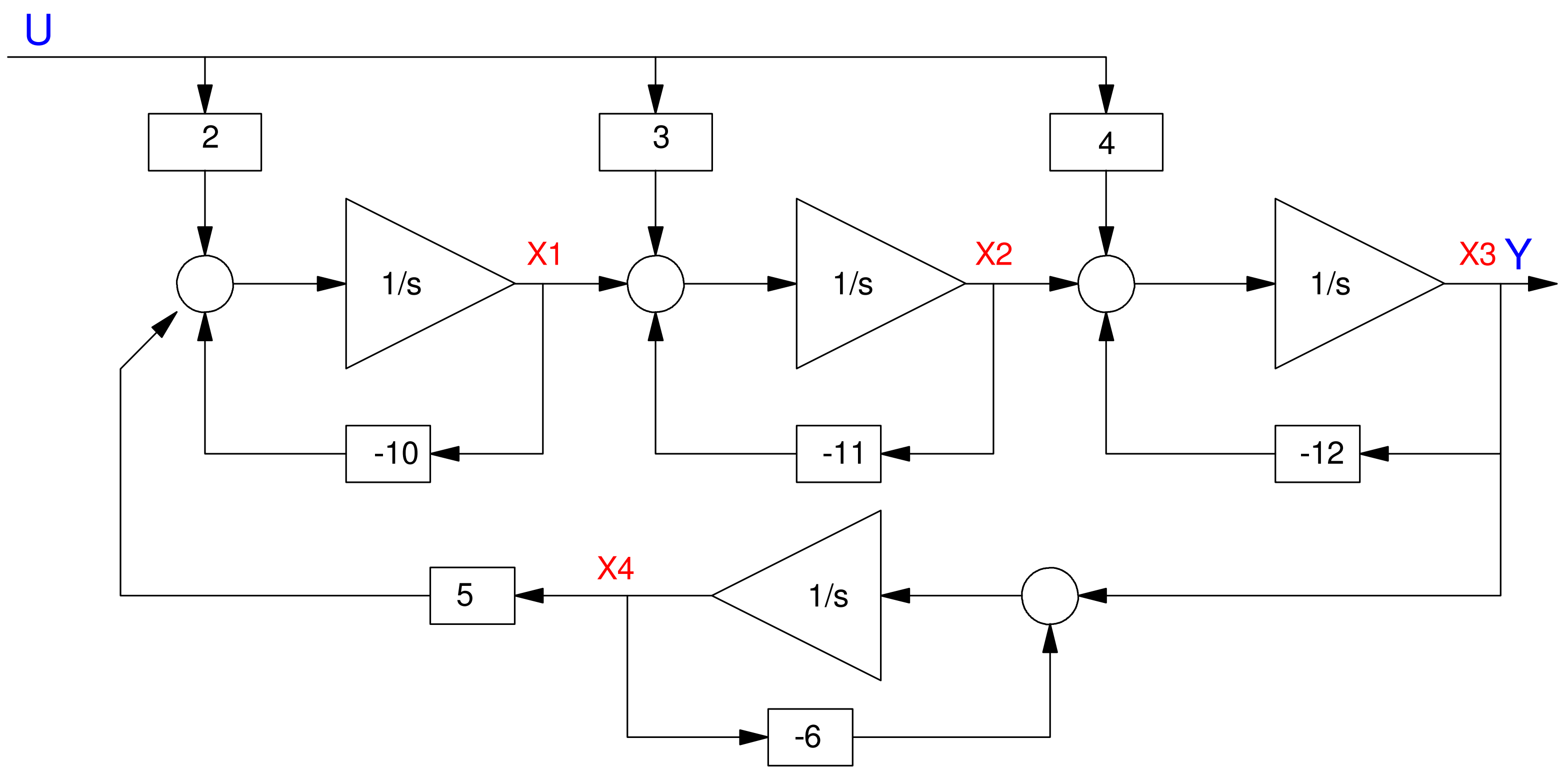
$$Y = CX + DU$$

- Matrix representation
- Foundation of Modern Control
- X defines the energy in the system
- sX defines how the energy moves around in the system
- Y is what you measure

Example 1: Bock Diagram to State Space

Give the state-space model for the following system

Find the transfer function from U to Y



Solution: Write the differential equations

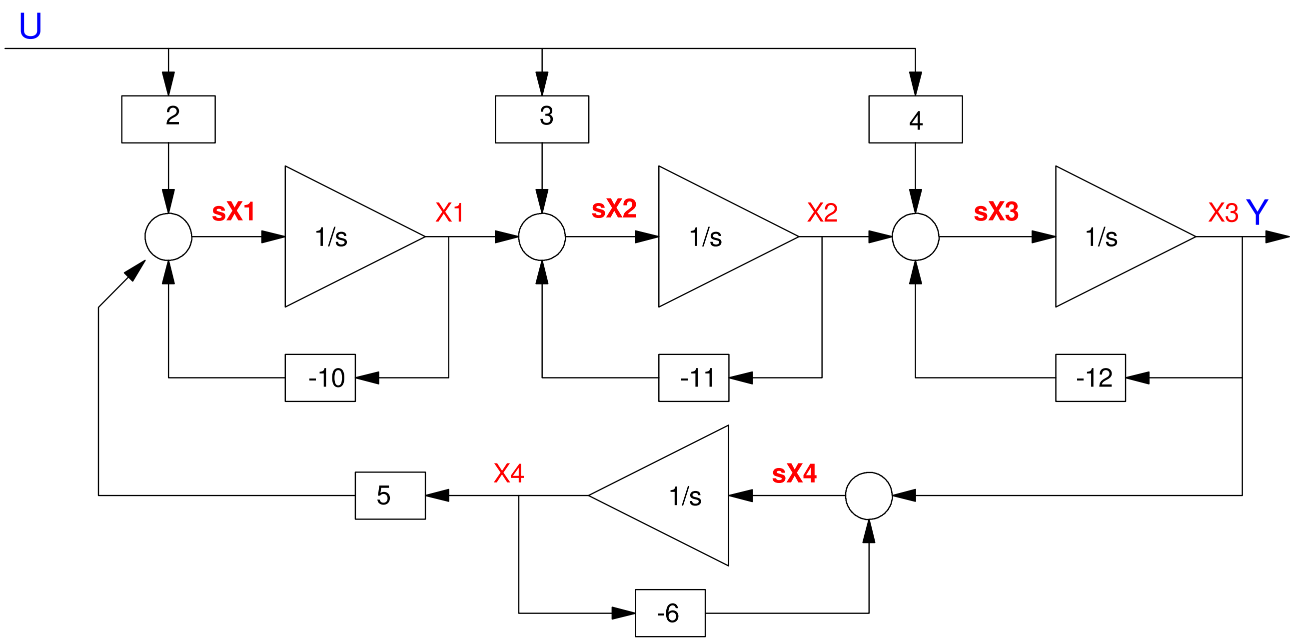
$$sX_1 = 2U - 10X_1 + 5X_4$$

$$sX_2 = 3U + X_1 - 11X_2$$

$$sX_3 = 4U + X_2 - 12X_3$$

$$sX_4 = X_3 - 6X_4$$

$$Y = X_3$$

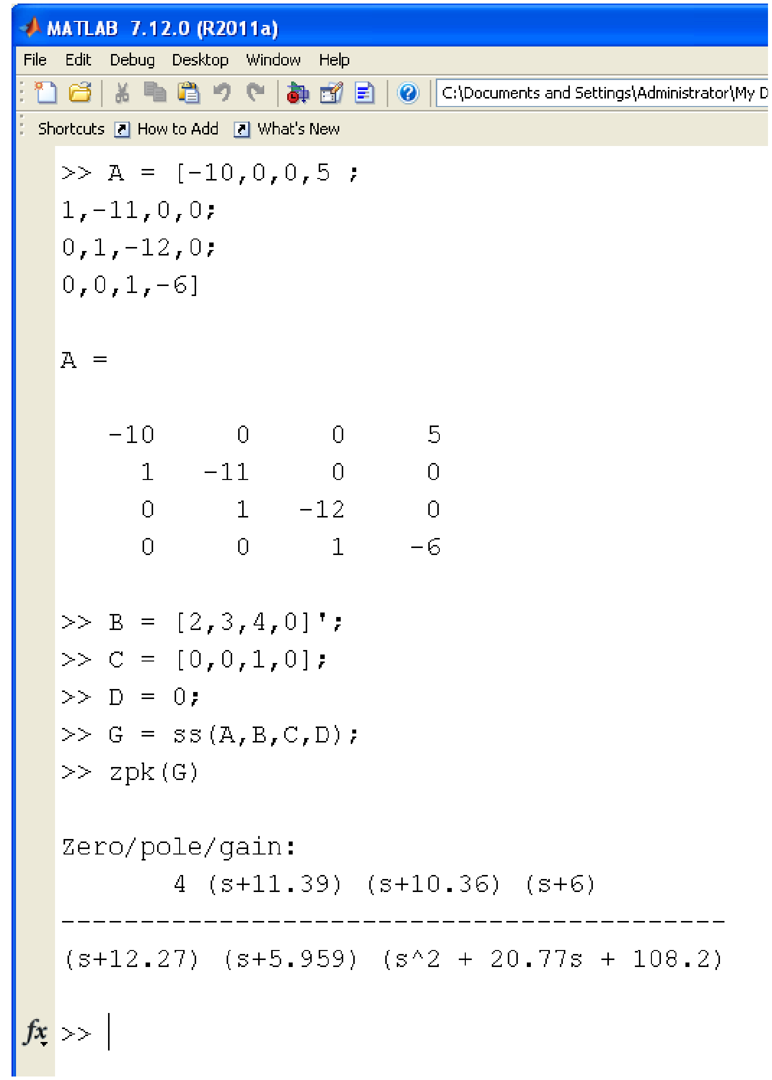


Place in matrix form

$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 & 5 \\ 1 & -11 & 0 & 0 \\ 0 & 1 & -12 & 0 \\ 0 & 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0]U$$

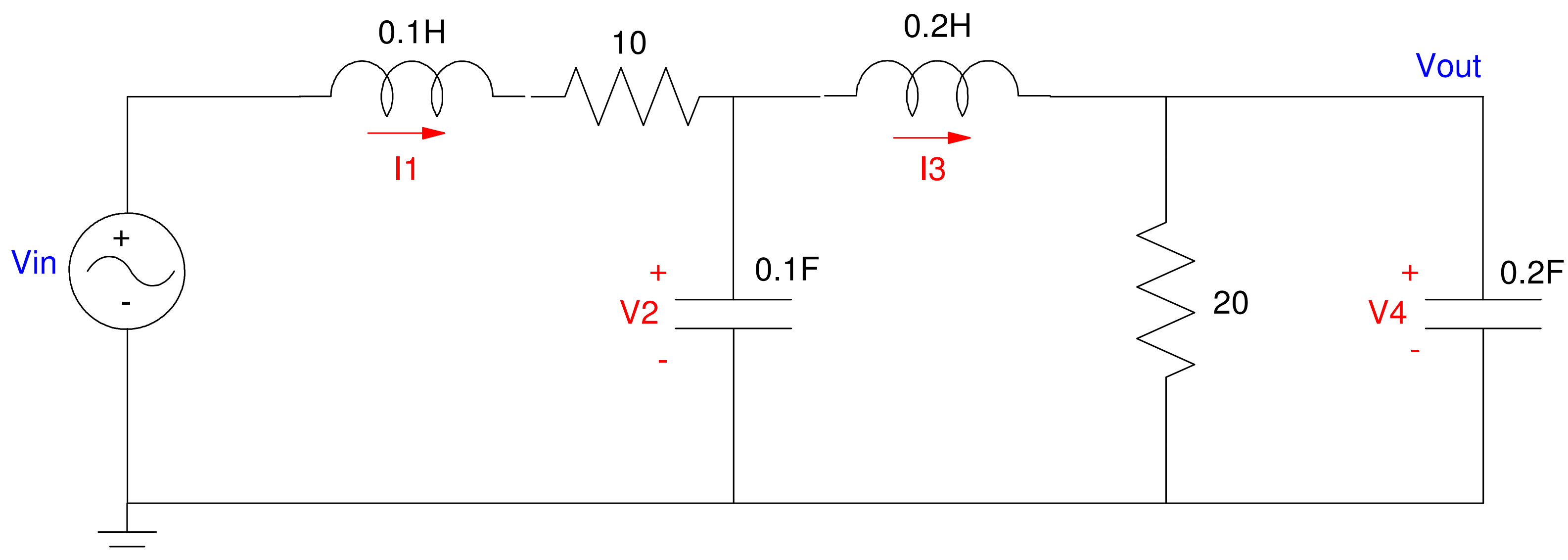
Use Matlab to find the transfer function



Example 2: RLC Circuit

For the following RLC circuit,

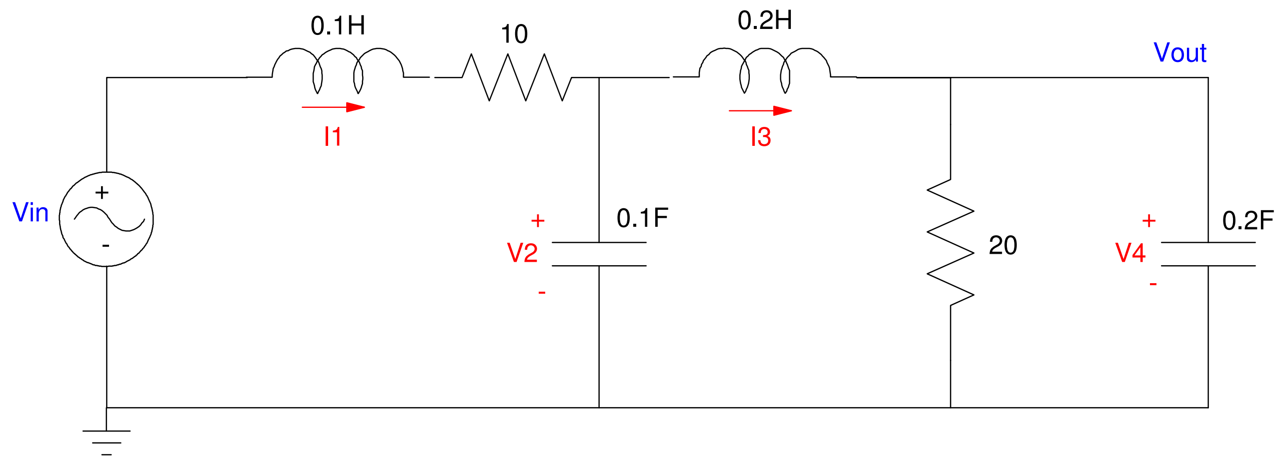
- Find the transfer function for the following circuit from Vin to Vout
- Find the dominant pole of this system
- Determine a 1st or 2nd-order approximation for this system:



Step 1: Define the states (X)

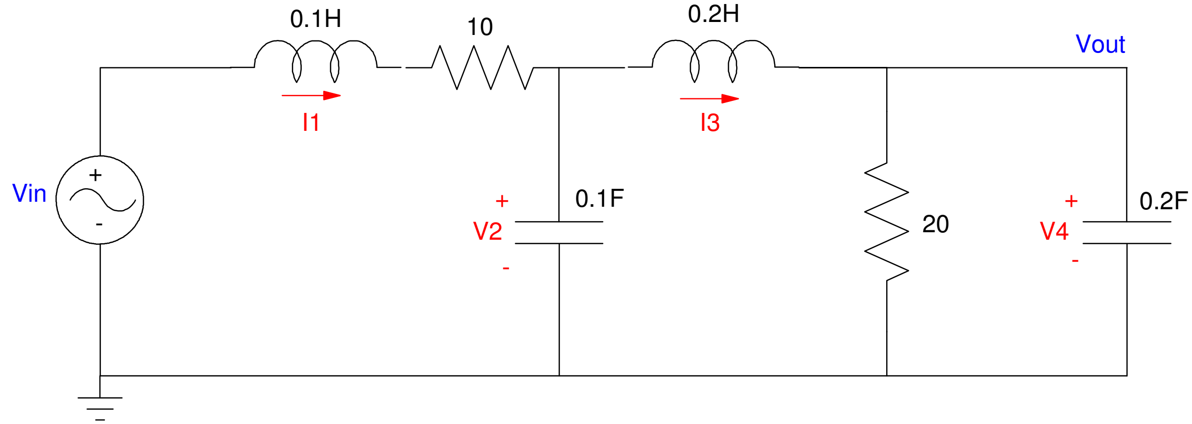
- Defines the energy in the system
- One option is

$$X = \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}$$



Step 2: Define how the energy moves around

- Determine the dynamics



Trick for circuits:

$$V = L\frac{dI}{dt}$$

$$I = C\frac{dV}{dt}$$

For each inductor and capacitor:

$$V_{L1} = 0.1sI_1 = V_{in} - (V_2 + 10I_1)$$

$$I_{C2} = 0.1sV_2 = I_1 - I_3$$

$$V_{L3} = 0.2sI_3 = V_2 - V_4$$

$$I_{C4} = 0.2sV_4 = I_3 - \frac{V_4}{20}$$

Step 3: Group terms

$$sI_1 = 10V_{in} - 10V_2 - 100I_1$$

$$sV_2 = 10I_1 - 10I_3$$

$$sI_3 = 5V_2 - 5V_4$$

$$sV_4 = 5I_3 - 0.25V_4$$

Place in matrix form (i.e. state-space)

$$\begin{bmatrix} sI_1 \\ sV_2 \\ sI_3 \\ sV_4 \end{bmatrix} = \begin{bmatrix} -100 & -10 & 0 & 0 \\ 10 & 0 & -10 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 5 & -0.25 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$V_{out} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + [0]V_{in}$$

Now, in Matlab, you can find the transfer function:

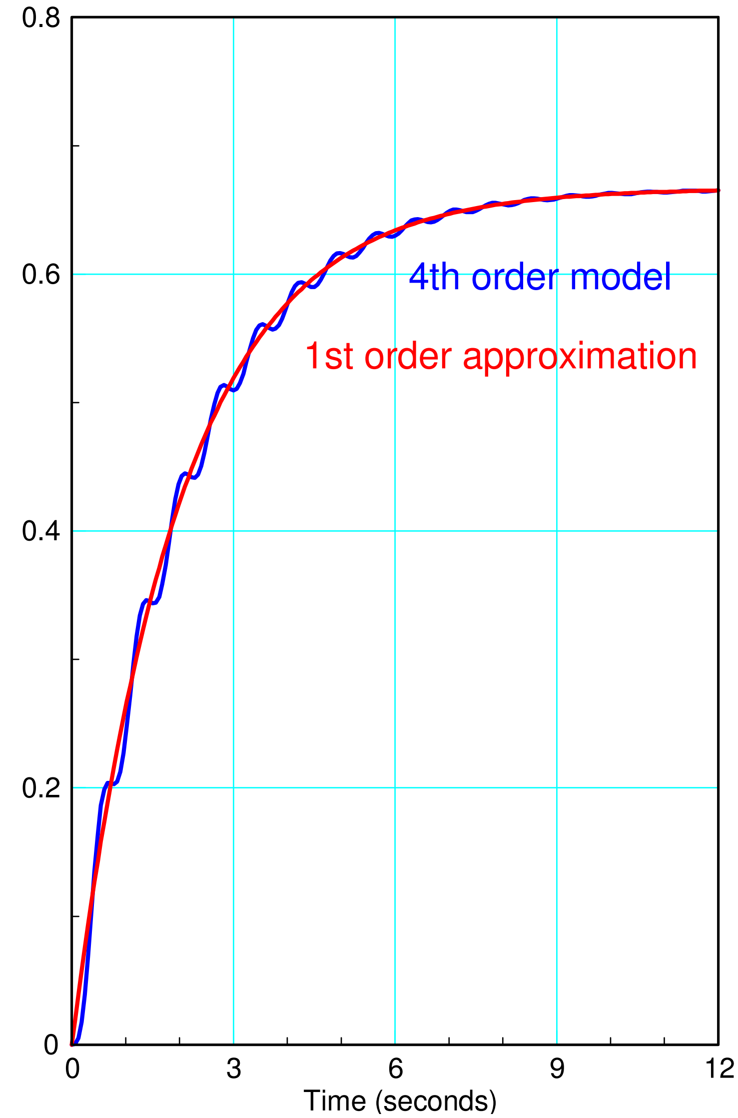
```
A = [-100,-10,0,0;10,0,-10,0;0,5,0,-5;0,0,5,-0.25]

B = [10;0;0;0]

C = [0,0,0,1]

D = 0

G4 = ss(A,B,C,D);

zpk(G4)
```

```
                    2500
--------------------------------------------
(s+98.99) (s+0.5025) (s^2 + 0.7525s + 75.38)
```

Its first-order approximation is:

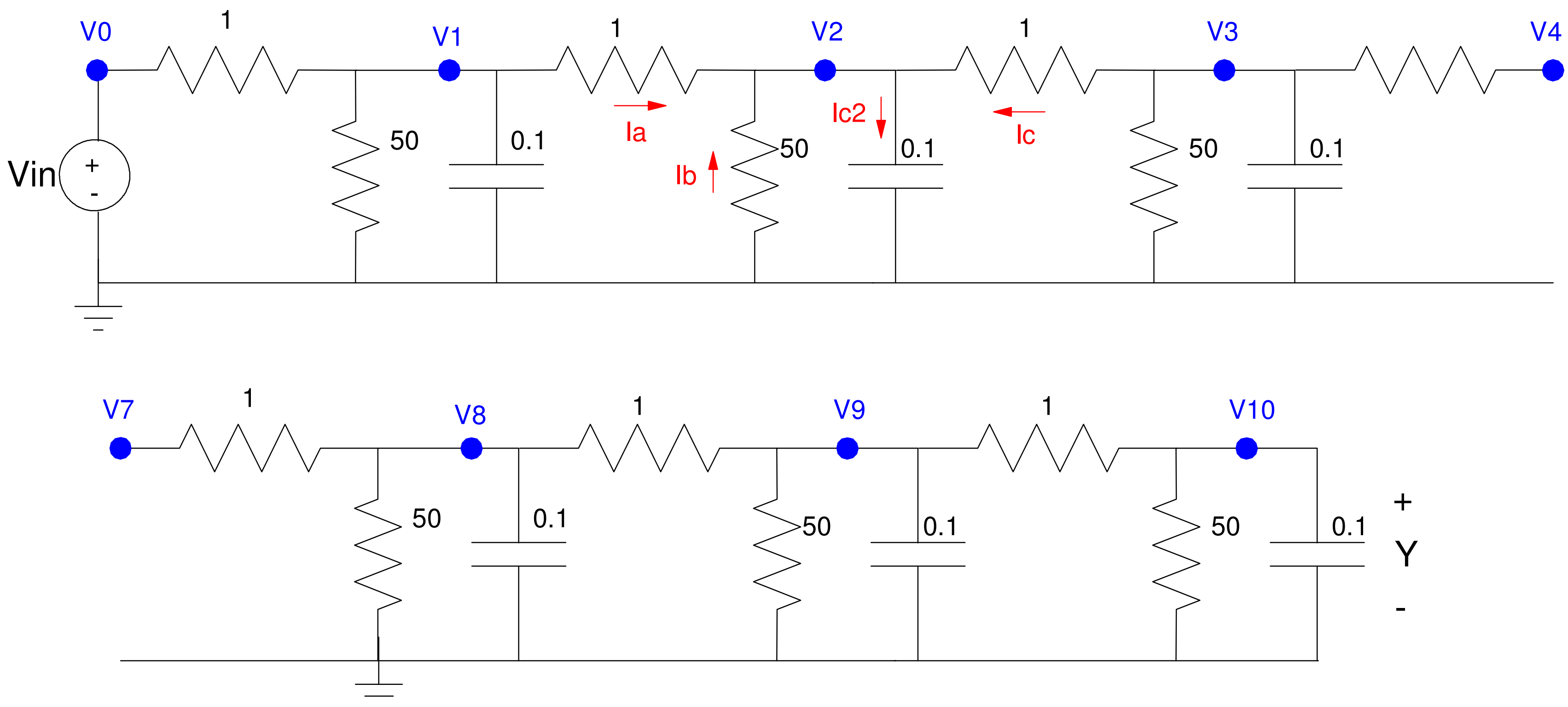
- A 1st-order system
- With a DC gain of 0.6667, and
- Ts = 8 seconds (4 / 0.5025).

```
DC = evalfr(G4,0)

    0.6667

G1 = tf(0.6667*0.5025,[1,0.5025])


  0.335
----------
s + 0.5025

y4 = step(G4,t);
y1 = step(G1,t);
plot(t,y4,'b',t,y1,'r');
```



Example 3: RC Filter (Heat Equation)

Consider next the following 10-stage RC filter



Voltage Node Equation at V2

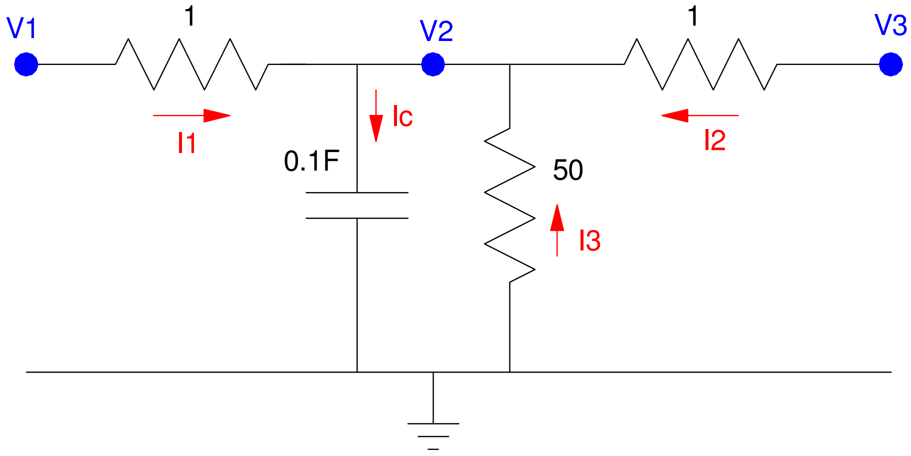
Current In = Current Out

$$I_c = I_1 + I_2 + I_3$$

$$CsV_2 = \left(\frac{V_1 - V_2}{1}\right) + \left(\frac{0 - V_2}{50}\right) + \left(\frac{V_3 - V_2}{1}\right)$$

$$sV_2 = 10V_1 - 20.2V_2 + 10V_3$$

This repeats for nodes 1..9.



Node #10 is slightly different

$$I_{c10} = CsV_{10} = \left(\frac{V_9 - V_{10}}{1}\right) + \left(\frac{0 - V_{10}}{50}\right)$$

$$sV_{10} = 10V_9 - 10.2V_{10}$$

In Matlab:

```
A = zeros(10,10);
for i=1:9
   A(i,i) = -20.2;
   A(i,i+1) = 10;
   A(i+1,i) = 10;
   end
A(10,10) = -10.2;
```

```
  -20.2000   10.0000         0         0         0         0         0         0         0         0
   10.0000  -20.2000   10.0000         0         0         0         0         0         0         0
         0   10.0000  -20.2000   10.0000         0         0         0         0         0         0
         0         0   10.0000  -20.2000   10.0000         0         0         0         0         0
         0         0         0   10.0000  -20.2000   10.0000         0         0         0         0
         0         0         0         0   10.0000  -20.2000   10.0000         0         0         0
         0         0         0         0         0   10.0000  -20.2000   10.0000         0         0
         0         0         0         0         0         0   10.0000  -20.2000   10.0000         0
         0         0         0         0         0         0         0   10.0000  -20.2000   10.0000
         0         0         0         0         0         0         0         0   10.0000  -10.2000
```

```
B = [10;0;0;0;0;0;0;0;0;0]
C = [0,0,0,0,0,0,0,0,0,1];
D = [0];
```

```
>> G = ss(A,B,C,D);
>> evalfr(G,0)

    0.4325

>> zpk(G)

                                                  10000000000
----------------------------------------------------------------------------------------------------------
(s+39.31) (s+36.72) (s+32.67) (s+27.51) (s+21.69) (s+15.75) (s+10.2) (s+5.539) (s+2.181) (s+0.4234)
```

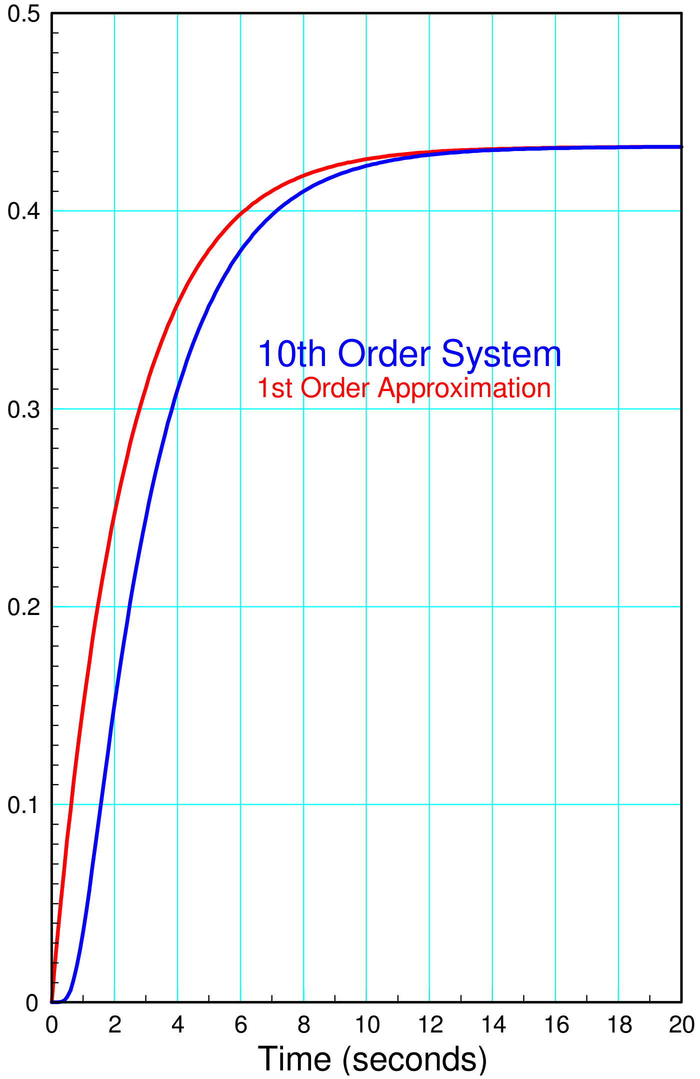
This is the transfer function for this RC filter. Its response should

- Have a DC gain of 0.4325, and
- A 2% settling time of 9.44 seconds (4 / 0.4234)

$$G(s) \approx \left(\frac{0.1793}{s+0.4234}\right)$$

The actual step-response from Matlab is

```
G1 = zpk([],[-0.4234],0.1793)

   0.1793
----------
(s+0.4234)

t = [0:0.001:10]';
y1 = step(G1,t);
y10 = step(G,t);
plot(t,y10,'b',t,y1,'r');
```

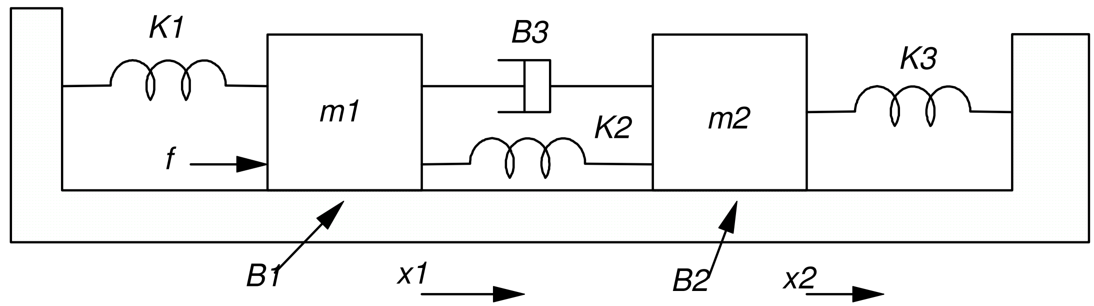


Example 4: Mass-Spring System

Use an electrical dual

$$I = \frac{1}{R} \cdot V$$

$$F = Ms^2 \cdot X$$



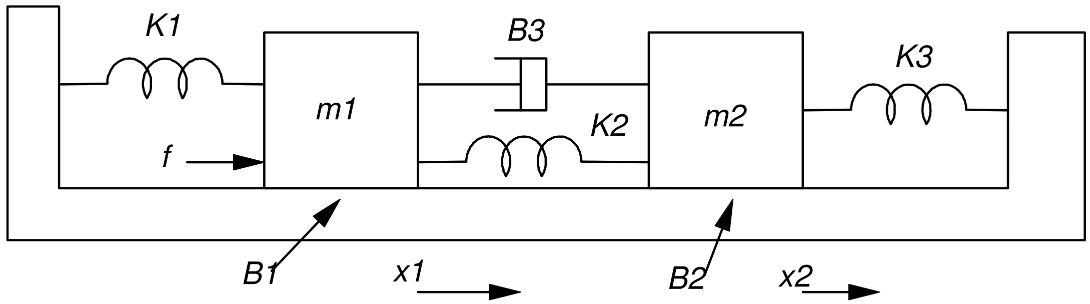
Current <> Force

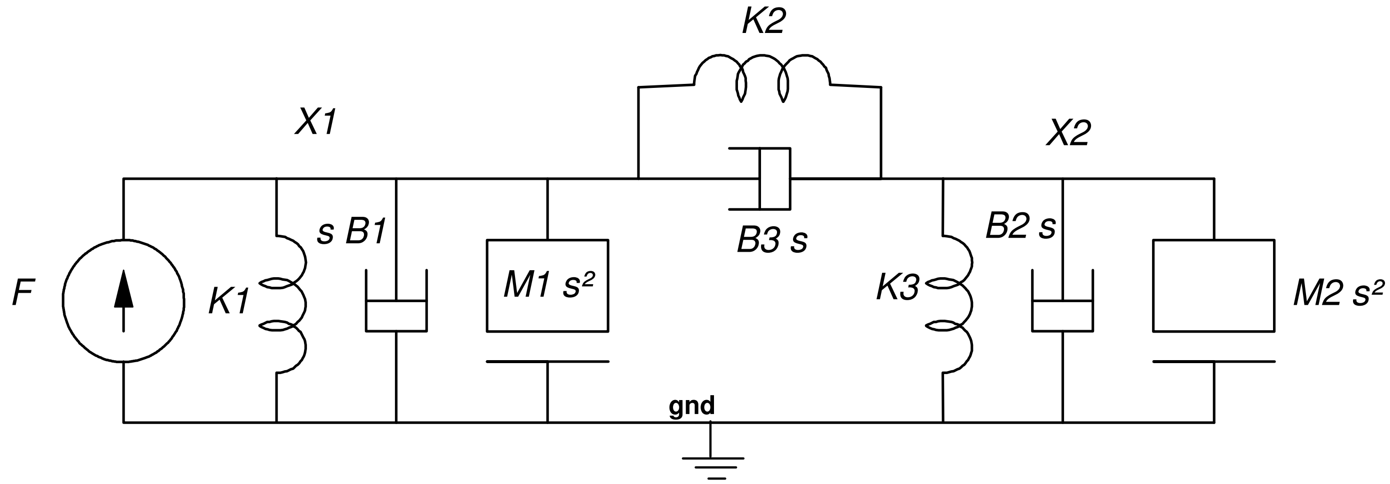
Voltage <> Position

Admittance

- Ms^2 (for a mass)
- Bs (for friction)
- K (for a spring)

Circuit Equivalent



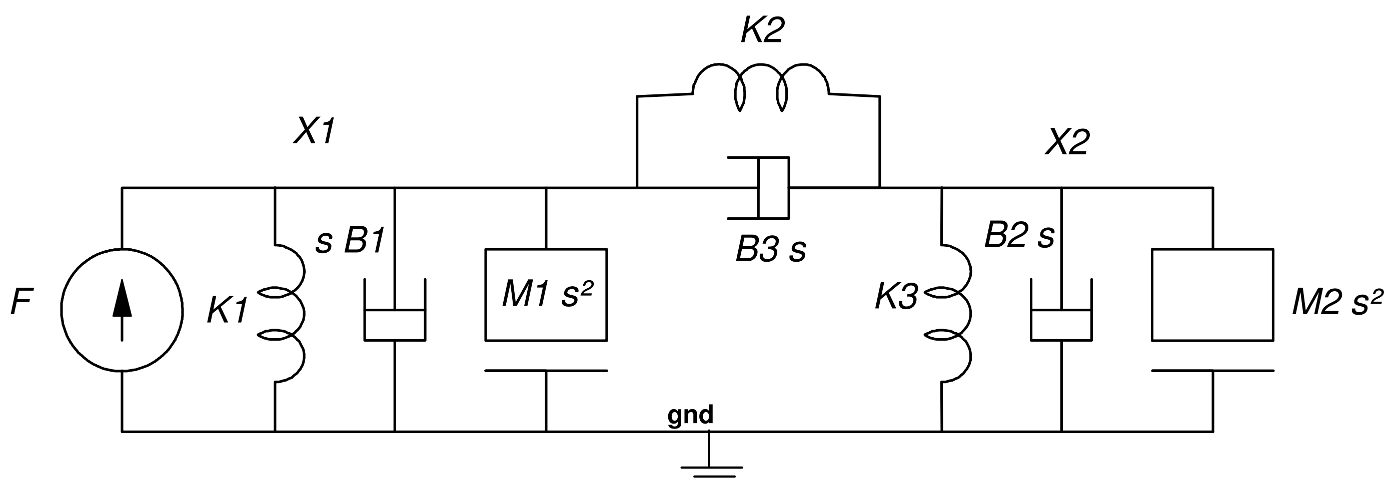


Dynamics

- Voltage Node Equations

$$(K_1 + B_1 s + M_1 s^2 + K_2 + B_3 s)X_1 - (K_2 + B_3 s)X_2 = F$$

$$(M_2 s^2 + B_2 s + K_3 + K_2 + B_3 s)X_2 - (K_2 + B_3 s)X_1 = 0$$



Solving for the highest derivative:

$$M_1 s^2 X_1 = -(K_1 + K_2 + B_1 s + B_3 s)X_1 + (K_2 + B_3 s)X_2 + F$$

$$M_2 s^2 X_2 = -(B_2 s + K_3 + K_2 + B_3 s)X_2 + (K_2 + B_3 s)X_1$$

Place in matrix (state-space) form

- Note: 2N states (N position states, N velocity states)
- Potential Energy & Kinetic Energy

$$s\begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ sX_1 \\ sX_2 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{-(K_1+K_2)}{M_1}\right) & \left(\frac{K_2}{M_1}\right) & \left(\frac{-(B_1+B_3)}{M_1}\right) & \left(\frac{B_3}{M_1}\right) \\ \left(\frac{K_2}{M_2}\right) & \left(\frac{-(K_2+K_3)}{M_2}\right) & \left(\frac{B_3}{M_2}\right) & \left(\frac{-(B_2+B_3)}{M_2}\right) \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ sX_1 \\ sX_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \left(\frac{1}{M_1}\right) \\ 0 \end{bmatrix} F$$

Finding the Transfer Function:

- M = 1kg, B = 2 Ns/m, K =10 N/m
- Y = X2

$$s\begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20 & 10 & -4 & 2 \\ 10 & -20 & 2 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} F$$

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} + [0]F$$

MATLAB Code:

The mass spring system has

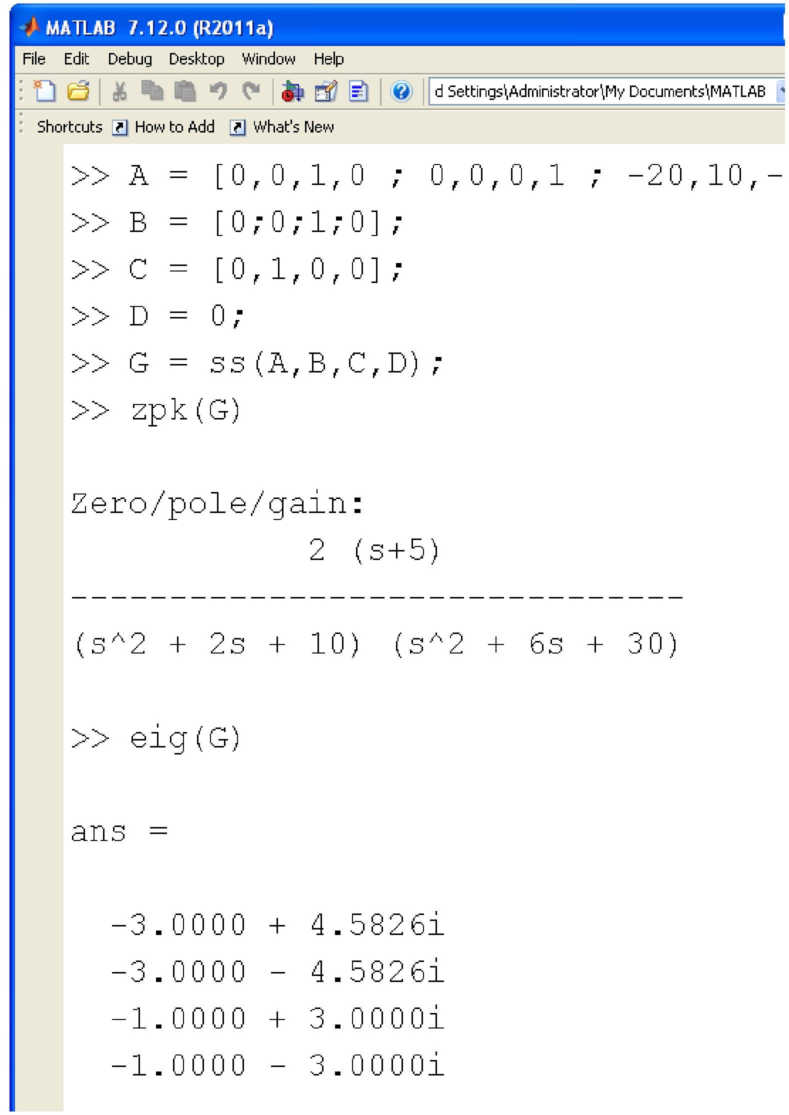
- Four poles
- One set is at {-1 +/- j3}
- The other set is at {-3 +/- j4.58}

The 2% settling time is 4 seconds

- Slowest pole is at s = -1 + jX

The slow pole oscillates at 3 rad/sec

- approx 1/2 Hz




```
>> A = [0,0,1,0 ; 0,0,0,1 ; -20,10,-
>> B = [0;0;1;0];
>> C = [0,1,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

Zero/pole/gain:
            2 (s+5)
-------------------------------
(s^2 + 2s + 10) (s^2 + 6s + 30)

>> eig(G)

ans =

  -3.0000 + 4.5826i
  -3.0000 - 4.5826i
  -1.0000 + 3.0000i
  -1.0000 - 3.0000i
```

Transfer Function

$$Y = \left(\frac{2(s+5)}{(s+1\pm j3)(s+3\pm j4.58)} \right) F$$

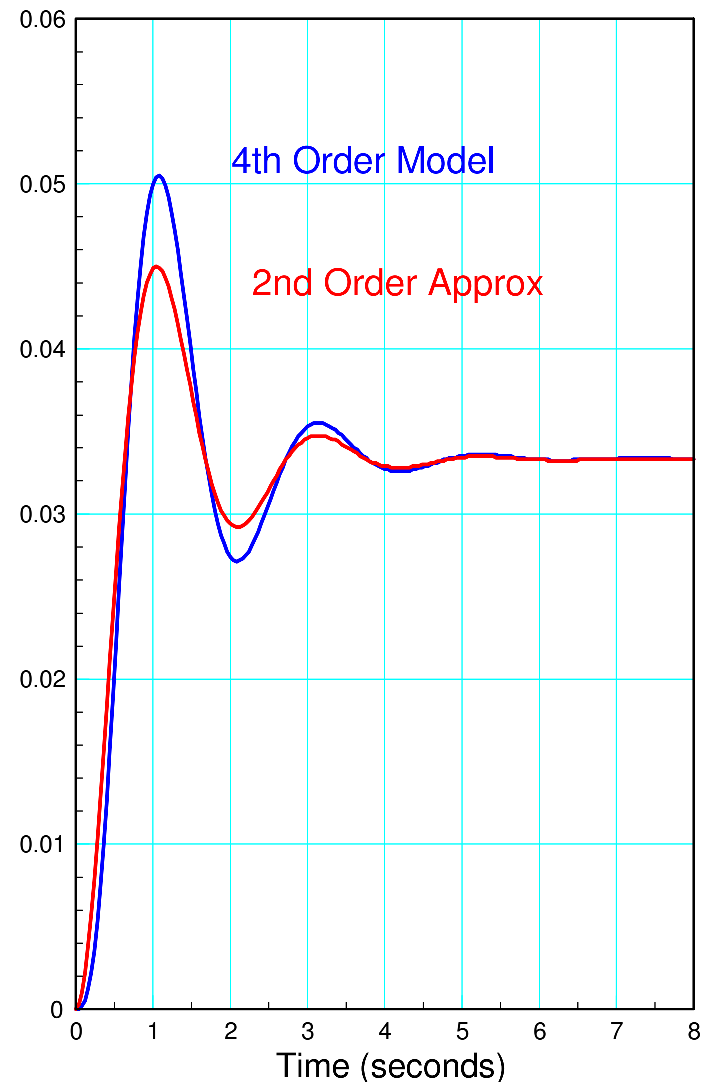
Second Order Approximation

- Keep the dominant pole (s = -1 + j3)
- Match the DC gain

$$Y \approx \left(\frac{0.333}{(s+1\pm j3)} \right) F$$

Plot the step response

```
t = [0:0.1:100]';
G2 = zpk([],[-1+j*3,-1-j*3],0.333);
y = step(G,t);
y2 = step(G2,t);
plot(t,y,t,y2,'r')
xlabel('Time (seconds)');
ylabel('X2 (meters)');
```



Summary

State-Space makes modeling RLC circuits much easier

- States = Energy
- Current in Inductors
- Voltage Across Capacitors

State-Space makes modeling mass-spring systems fairly easy

- Convert to the circuit equivalent
- Write the voltage node equations

With Matlab, finding the dynamics of 10th order systems is fairly easy

- What's most important is get the equations right
- Matlab will then find the resulting transfer funciton and step response