LaGrangian Formulation of System Dynamics

NDSU ECE 463/663

Lecture #6

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Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Finding the dynamics of a nonlinear system:

Circuit analysis tools work for simple lumped systems.

- RC Circuits
- RLC Circuits

For more complex systems, especially nonlinear ones, this approach fails.

The Lagrangian formulation for system dynamics is a way to deal with any system.

- It defines the energy in the system
- It then determines how the energy moves about the system
- The result is a tool that can be used to find the dynamics of linear *and* nonlienar systems

Definitions:

KE Kinetic Energy in the system

PE Potential Energy

$\frac{\partial}{\partial t}$ The partial derivative with respect to 't'.

$\frac{d}{dt}$ The full derivative with respect to t.

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \ldots$$

L Lagrangian = KE - PE

Partial vs. Full Derivatives

A full derivative includes partial derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \dots$$

When taking a parial derivative everything else is treated like a constant

$$\frac{\partial}{\partial t}(x^2y^3t^4) = (x^2y^3)(4t^3)$$

It doesn't matter that x and y are functions of t. That is taken into account in other terms in the full derivative

- If you took this into acount when taking the partial with respect to t, you'd double count these terms

Example: Let

$$x(t) = 2t^2 \qquad y(t) = \cos(3t) \qquad f = \sin(2x) \cdot y^2 \cdot t^3$$

Find

$$\frac{df}{dt} = \frac{d}{dt}(\sin(2x) \cdot y^2 \cdot t^3)$$

Solution

$$\frac{df}{dt} = \frac{\partial}{\partial x}(\sin(2x) \cdot y^2 \cdot t^3)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}(\sin(2x) \cdot y^2 \cdot t^3)\frac{\partial y}{\partial t} + \frac{\partial}{\partial t}(\sin(2x) \cdot y^2 \cdot t^3)\frac{\partial t}{\partial t}$$

$$\frac{df}{dt} = \left(2\cos(2x) \cdot y^2 \cdot t^3\right)\dot{x} + \left(t^3 \sin(2x) \cdot 2y\right)\dot{y} + \left(\sin(2x) \cdot y^2 \cdot 3t^2\right)$$

Often times, people forget the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ terms. You need them.

Procedure for LaGrangian Dynamics:

1) Define the kinetic and potential energy in the system.
2) Form the Lagrangian:

$$L = KE - PE$$

3) The input is then

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

where F_i is the input to state x_i. Note that

- If x_i is a position, F_i is a force.
- If x_i is an angle, F_i is a torque

Also pay attention to the full derivatives and the partial derivatives.

Example: Rocket Dynamics

Step 1: Determine the potential and kinetic energy

Potential Energy

$$PE = mgx$$

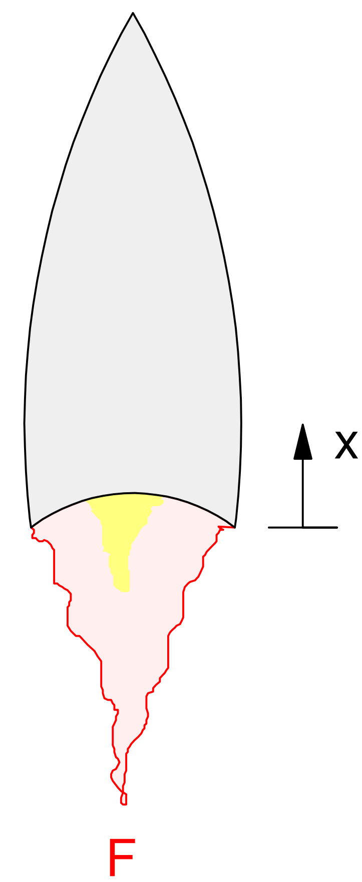
Kinetic Energy:

$$KE = \frac{1}{2}m\dot{x}^2$$

Step 2: Set up the LaGrangian

$$L = KE - PE$$

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$



Step 3: Take the partials

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}(m\dot{x}) - (-mg)$$

Take the full derivative with respect to t

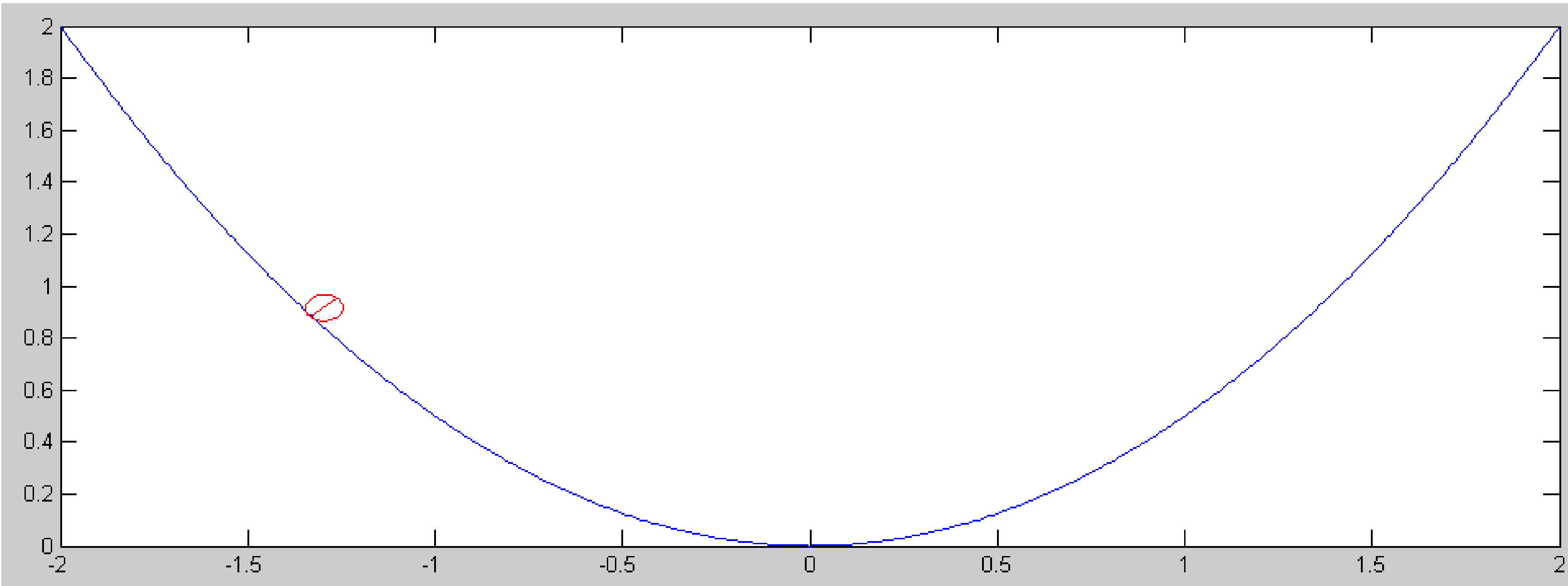
$$\boxed{F = m\ddot{x} + \dot{m}\dot{x} + mg}$$

Note that if the rocket is loosing mass you get the term $\dot{m}\dot{x}$. If you leave this term out, the rocket misses the target.

Example 2: Ball in a parabolic bowl

Determine the dynamics of a ball rolling in a bowl characterized by

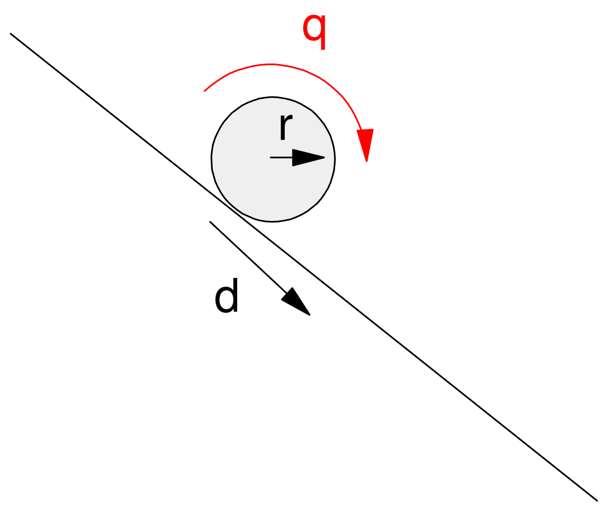
$$y = \frac{1}{2}x^2$$



Step 1: Define the kinetic and potential energy

Potential Energy:

$$PE = mgy = \frac{1}{2}mgx^2$$



Kinetic Energy: This has two terms, one for translation and one for rotation .

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}J\dot{\theta}^2$$

The velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

The rotational velocity is

$$d = r\theta$$

$$v = r\dot{\theta}$$

Note that

$$y = \tfrac{1}{2}x^2$$

$$\dot{y} = x\dot{x}$$

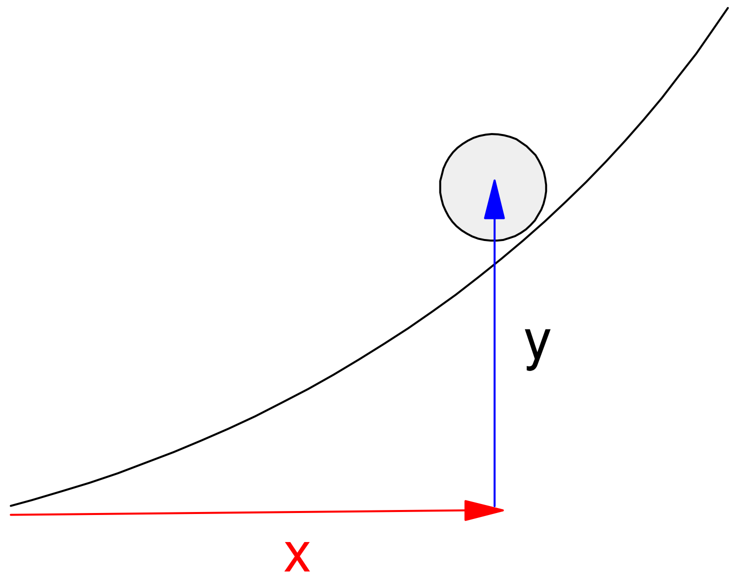
gives

$$KE = \tfrac{1}{2}mv^2 + \tfrac{1}{2}J(\tfrac{v}{r})^2$$

$$KE = \tfrac{1}{2}\left(m + \frac{J}{r^2}\right)v^2$$

$$KE = \tfrac{1}{2}\left(m + \frac{J}{r^2}\right)\left(\dot{x}^2 + \dot{y}^2\right)$$

$$KE = \tfrac{1}{2}\left(m + \frac{J}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$



The inertia depends upon what type of ball you are using:

$J = 0$	point mass with all the mass in the center
$J = \frac{2}{5}mr^2$	solid sphere
$J = \frac{2}{3}mr^2$	hollow sphere
$J = mr^2$	hollow cyllinder

Assume the ball is a solid sphere

$$KE = \frac{1}{2}\left(m + \frac{\frac{2}{5}mr^2}{r^2}\right)\left(\dot{x}^2 + (x\dot{x})^2\right)$$

$$KE = 0.7m\left(1^2 + x^2\right)\dot{x}^2$$

Step 2: Form the LaGrangian

$$L = KE - PE$$

$$L = 0.7m\left(1^2 + x^2\right)\dot{x}^2 - \tfrac{1}{2}mgx^2$$

Step 3: Take derivatives (parial and full)

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

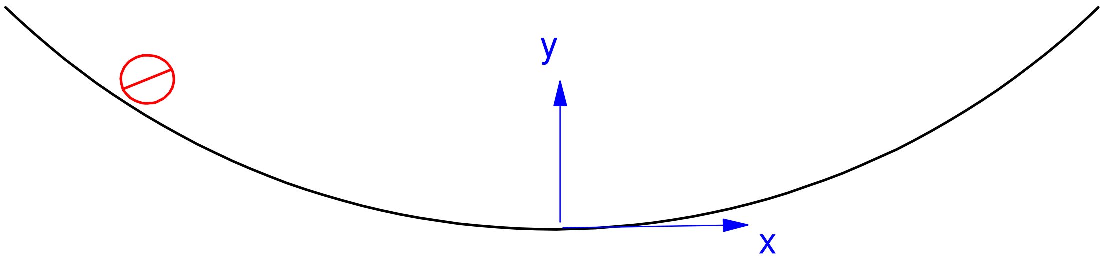
$$F = \frac{d}{dt}\left(1.4m\left(1 + x^2\right)\dot{x}\right) - \left(1.4mx\dot{x}^2 - mgx\right)$$

$$F = 1.4m(2x\dot{x})\dot{x} + 1.4m\left(1 + x^2\right)\ddot{x} - \left(1.4mx\dot{x}^2 - mgx\right)$$

$$\boxed{F = 1.4mx\dot{x}^2 + 1.4m\left(1 + x^2\right)\ddot{x} + mgx}$$

Assuming m = 1, F = 0

$$\boxed{\ddot{x} = -\left(\frac{\left(1.4\dot{x}^2+g\right)x}{1.4\left(1^2+x^2\right)}\right)}$$



Matlab Code (Ball.m)

```
while(t < 100)

ddx = -( 1.4*dx*dx + 9.8) * x / ( 1.4*(1 + x*x) );

% integrate

x = x + dx*dt;
dx = dx + ddx*dt;

% display the ball

:
:
```

Animation Trick

Drawing a line through the ball to show it rotating helps the animation

To draw this line, you need to know how far the ball has rolled:

$$l = r\theta$$

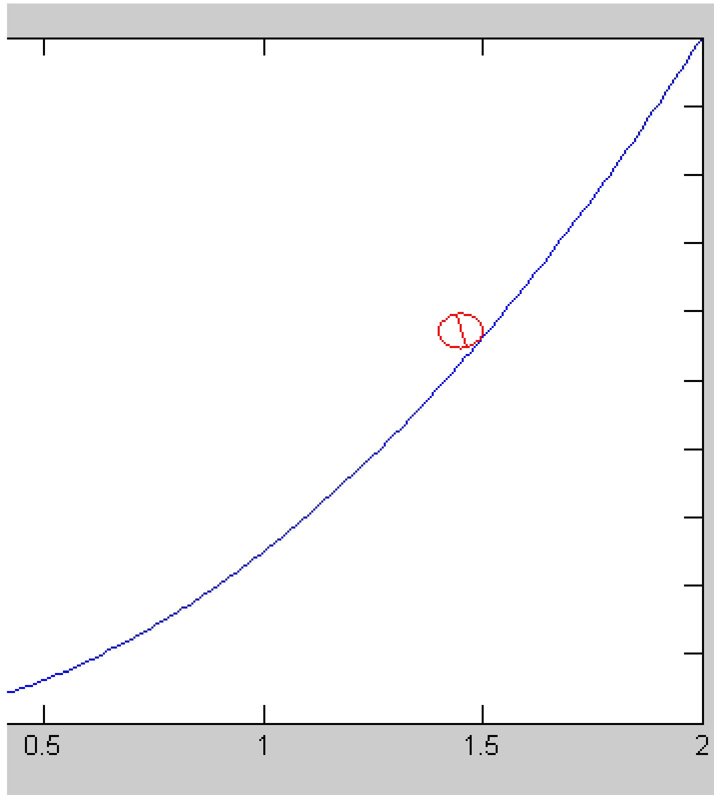
$$\sqrt{(dx)^2 + (dy)^2} = r \cdot d\theta$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = r \cdot d\theta$$

$$\theta = \frac{1}{r}\int \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

Since $y = \frac{1}{2}x^2$

$$\theta = \frac{1}{r}\int \left(\sqrt{1 + x^2}\right) dx$$



What is the frequency of oscillation for x small?

If x is small,

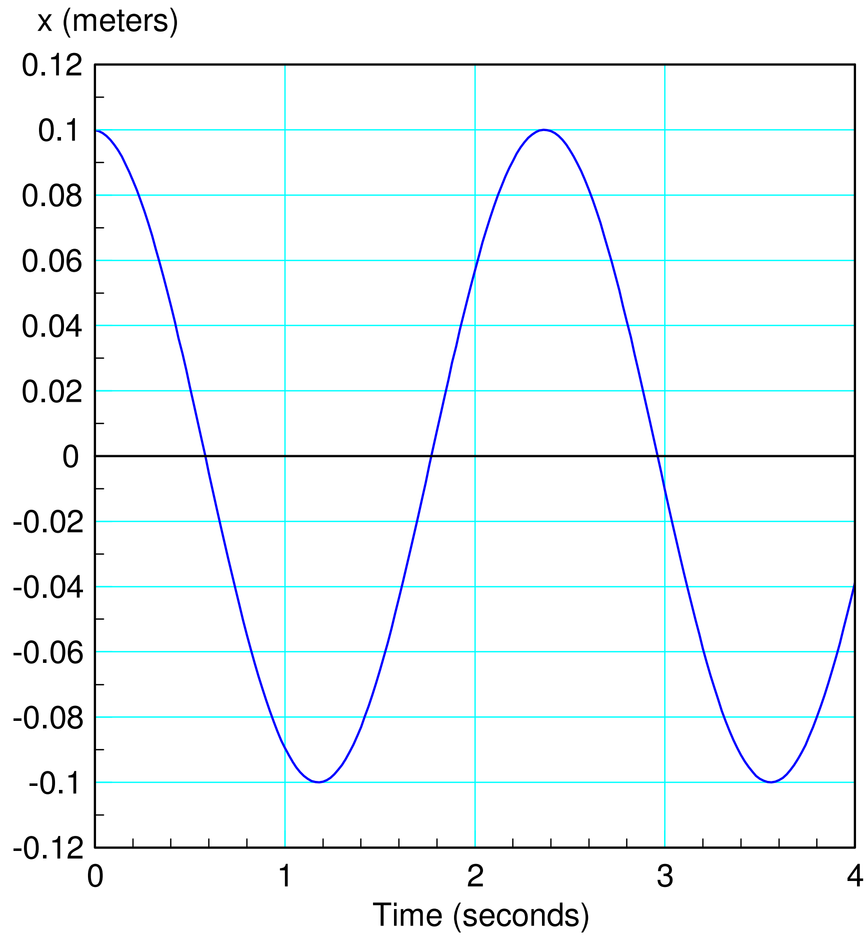
$$\dot{x}^2 x \approx 0 \qquad x^2 \approx 0$$

$$\ddot{x} = -\left(\frac{\left(1.4\dot{x}^2+g\right)x}{1.4\left(1^2+x^2\right)}\right) \approx -\left(\frac{g}{1.4}\right)x$$

$$\left(s^2 + \frac{g}{1.4}\right)x = 0$$

$$s = \pm j\sqrt{\frac{g}{1.4}} = j2.646$$

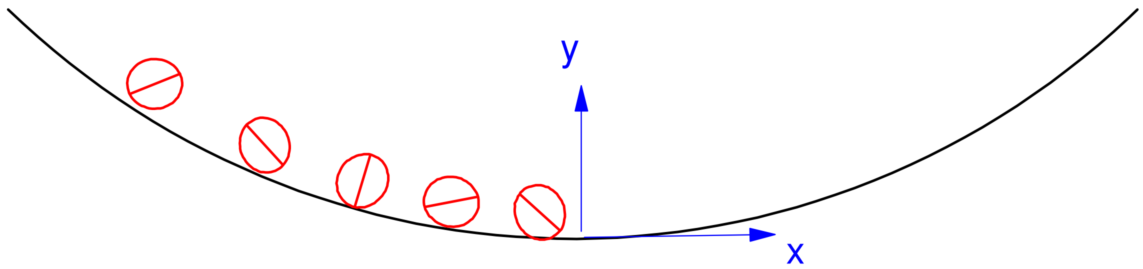
$$\text{Period} = \left(\frac{2\pi}{2.646}\right) = 2.375 \text{ sec}$$



Sidelight: Is the frequency of oscillation constant?

For a constant frequency of oscillation, you need

$$\ddot{x} = \text{constant}$$



Here, you have

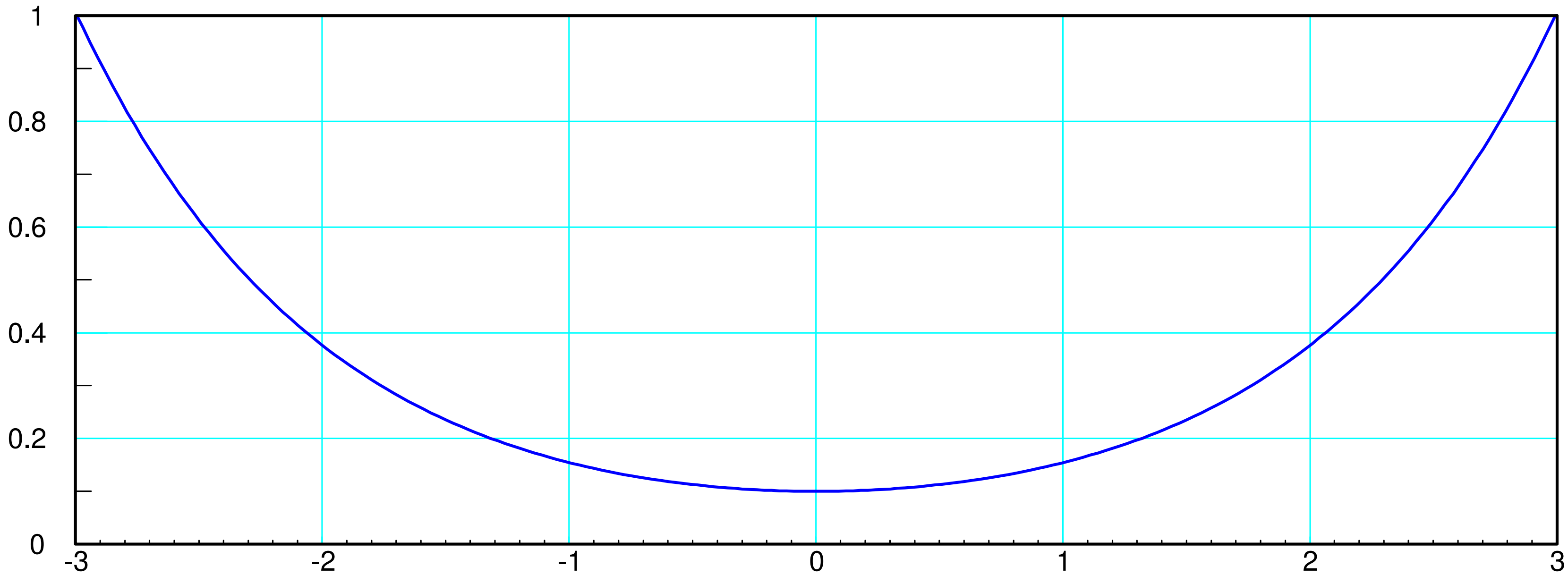
$$\ddot{x} = -\left(\frac{\left(1.4\dot{x}^2+g\right)x}{1.4\left(1^2+x^2\right)}\right)$$

No, it isn't constant. (i.e. this would make a bad clock)

What shape results in a constant frequency of oscillation?

Answer: A catenoid

$$y = a\ \cosh\left(\frac{x-b}{c}\right)$$



History of Clocks

Clocks are important for navigation

- They tell you your east / west longidude
- Old maps were accurate north / south, East / West were sketchy

A ship's chronometer (clock) was one of the most important instruments on the entire ship

- It told you your location east / west
- It was kept under lock and key in the Captain's quarters
- Only the Captain could touch it
- This was cutting edge technology in 1700

Railroads and Clocks

When Europe was connected with rail lines, it became important to have a univesal time

- Previously, each city kept its own time

This led to Europe adopting a common time

Railroad time clocks were the most accurate in the world

- Only one train can be on a track at any one time

Time Today: NIST

- One second is the duration of 9,192,631,770 cycles of the radiation associated with a specified transition of the cesium atom.
- https://www.nist.gov/system/files/documents/calibrations/sp432-02.pdf