Controllability and Observability

NDSU ECE 463/663

Lecture #11

Inst: Jake Glower

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Controllability and Observability

Controllability: The ability to drive a system to an arbitrary state in finite time.

Observability: The ability to determine the system's state from the output and its derivatives.

If a system is controllable, it means you can stabilize it.

- You can place the poles anywhere you like using full-state feedback
- ECE 463 Week 6 - 7

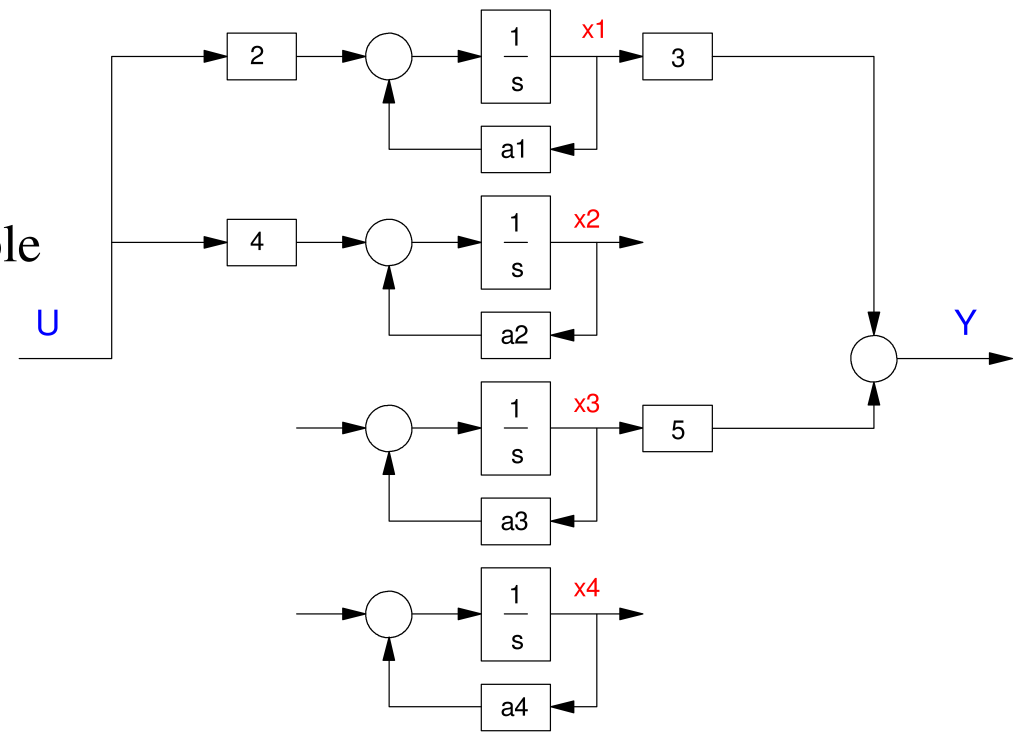
If a system is observable, it means you can build a state estimator

- You can build a full-order observer
- ECE 463 week 8 - 9

Controllable & Observabile

Example in Jordan form:

- x1 is controllable and Observable
- x2 is controllable but not observable
- x3 is obserevable but not controllable
- x4 is neither controllable nor observable



$$sX = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_f \end{bmatrix} X + \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 0 & 5 & 0 \end{bmatrix} X$$

Controllability & Observability by Inspection

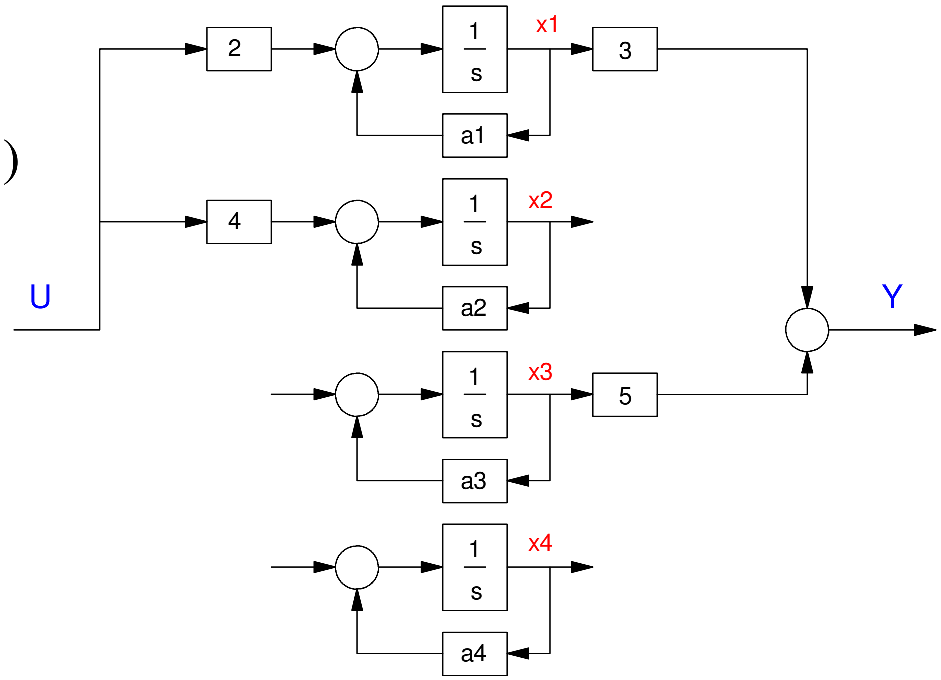
- If there is no pole / zero cancellation, a system is both controllable and observable.
- If the transfer function is the same order as the system, the system is both controllable and observable.

Example:

- The previous system is 4th order (4 states)
- The transfer function is 1st order

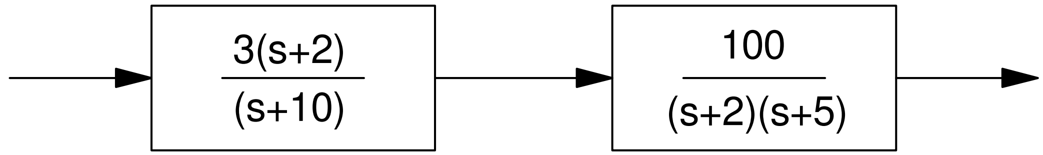
$$Y = \left(\frac{6}{s+a_1}\right) U$$

The system is either uncontrollable, unobservable, or both.



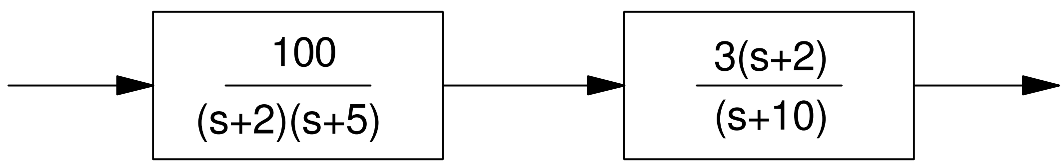
Pole Zero Cancellation

If a zero which cancels a pole comes before the pole, that state is uncontrollable.



The pole at -2 is uncontrollable due to zero canceling the pole before the pole

If a zero which cancels a pole comes after the pole, that state is unobservable.



The pole at -2 is unobservable due to zero canceling the pole after the pole

Controllability Matrix:

Theorem: If

$$rank\left(\left[\ B\ AB\ A^2B\ \ldots\ A^{N-1}B\ \right]\right) = N$$

the system is controllable.

Proof: This is easier to see in discrete time. Assume you have a 4th-order discrete-time system:

$$zX = AX + BU$$

If the input, U, can drive the system to any arbitrary state, then the system is controllable.

Assume X(0) = 0. X at k = 4 is then

$$X(0) = X_0$$

$$X_1 = AX_0 + BU_0$$

$$X_2 = AX_1 + BU_1 = A^2 X_0 + ABU_0 + BU_1$$

$$X_3 = AX_2 + BU_2 = A^3 X_0 + A^2 BU_0 + ABU_1 + BU_2$$

$$X_4 = AX_3 + BU_3 = A^4 X_0 + A^3 BU_0 + A^2 BU_1 + ABU_2 + BU_3$$

or in matrix form

$$X_4 - A^4 X_0 = \begin{bmatrix} B & AB & A^2 B & A^3 B \end{bmatrix} \begin{bmatrix} U_3 \\ U_2 \\ U_1 \\ U_0 \end{bmatrix}$$

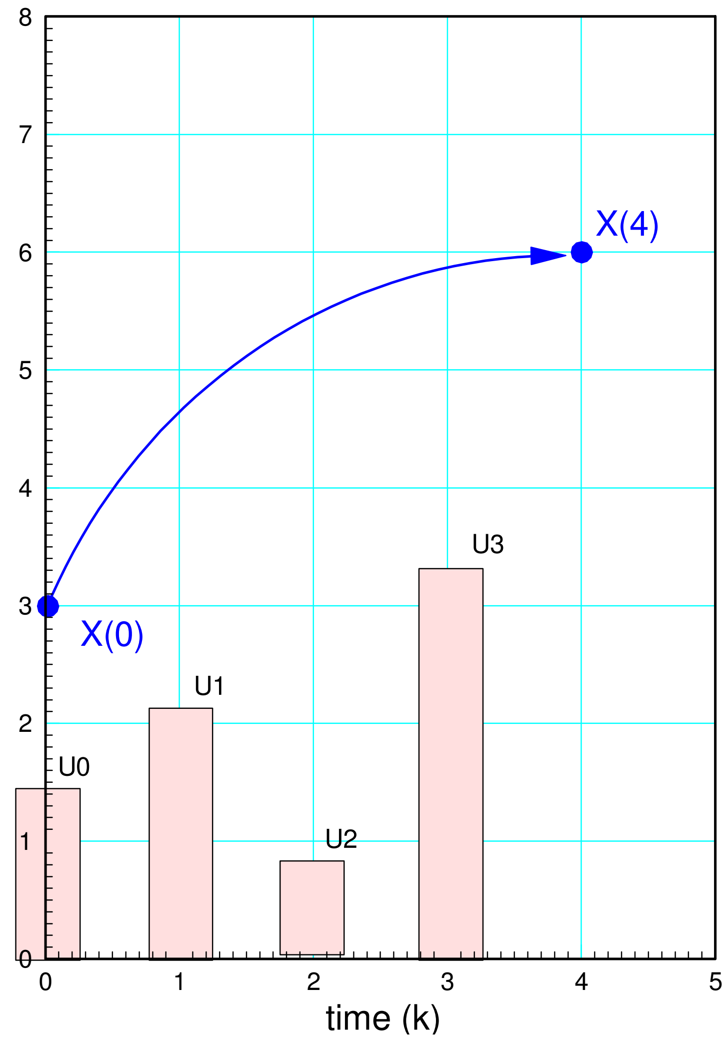
If the matrix (termed the *controllability matrix)* is full rank

$$\rho\left[\, B \; AB \; A^2B \; A^3B \,\right] = 4$$

then you can solve for U0..U3.

Translation:

- You can drive the system to any arbitrary state at k=4 using in puts {U0..U3}
- The system is controllable.



Calley Hamilton Theorem

How many terms do you need to include in the controllability matrix?

$$C = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots \end{bmatrix}$$

The Calley Hamilton theorem states that any matrix satisfies its own characteristic equation. (i.e. denominator polynomial.)

Example:

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

    -2      1      0      0
     1     -2      1      0
     0      1     -2      1
     0      0      1     -1

P = poly(eig(A))

P =    1.0000    7.0000   15.0000   10.0000    1.0000
```

The characteristic equation of A is

$$s^4 + 7s^3 + 15s^2 + 10s + 1 = 0$$

The Calley Hamilton theorem states that any matrix satisfies its own characteristic equtation:

$$A^4 + 7A^3 + 15A^2 + 10A + I = 0$$

Checking in Matlab:

```
A^4 + 7*A^3 + 15*A^2 + 10*A + 1*A^0

     0     0     0     0
     0     0     0     0
     0     0     0     0
     0     0     0     0
```

Translation:

- A^4 is linearly dependent upon $\{ I, \ A, \ A^2, \ A^3 \}$
- So is A^5 and so on
- Adding A^4 won't change the rank of the controllability matrix

When forming the controllability matrix for an Nth order system, you can stop after adding N terms. Hence

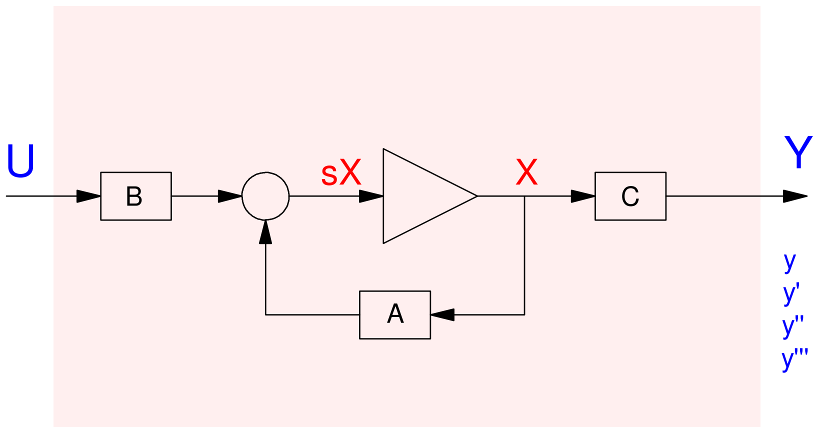
An Nth order system is controllable if

$$rank\left(\left[\ B \ AB \ A^2B \ \ldots \ A^{N-1}B \ \right]\right) = N$$

Observability

Observable:

- You can determine what the states are just using data from the input, the output, and their derivatives.
- This will be important later on in the course when we start designing state estimators:



X = ?

Observability Matrix

If the observability matrix is full rank, then the system is observable

$$\rho\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^{N-1} \end{bmatrix}\right) = N$$

- If a system is observable, you can determine the systems states from the system's output
- If a system is not observable, then at least one state does not contribute to the output you're looking at. This means there is no information about that state in the output and you cannot determine the value of that state.

Proof: Assume a 4th order system with U=0

$$sX = AX + BU$$

$$y = CX$$

The derivatives of y are

$$sy = s(CX) = CAX$$

$$s^2 y = s(CAX) = CA^2 X$$

$$s^3 y = s(CA^2 X) = CA^3 X$$

Putting this in matrix form

$$\begin{bmatrix} y \\ sy \\ s^2 y \\ s^3 y \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} X$$

If the observability matrix is full rank:

$$rank\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 4$$

then you can solve for X given only the output, y, and its derivatives.

- This means the system is observable.

Again, the Calley Hamilton theorem tells you that you can stop at A^{N-1}

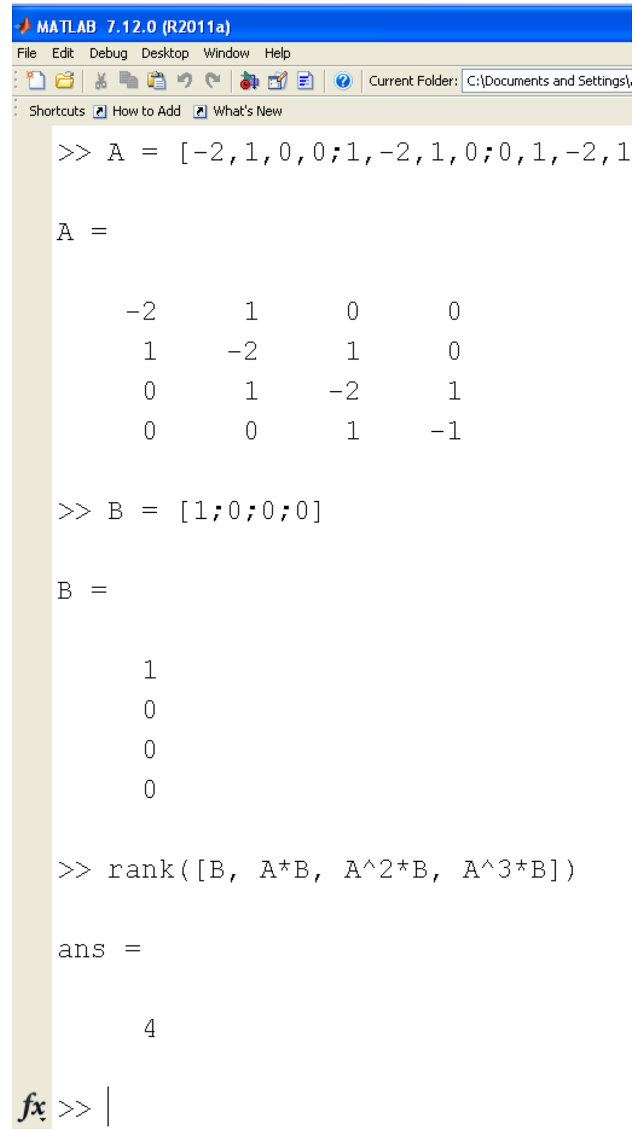
Rank, Determinant, and Eigenvalues

There are several ways to determine if a matrix is full rank in Matlab.

Rank(): The Matlab command *rank* determines how many linearly independent columns a matrix has. This is a Boolean test: each eigenvector either is or is not excited.

Example: 4-Stage RC filter

- The rank of the controllability matrix is 4
- A 4-stage RC filter is controllable



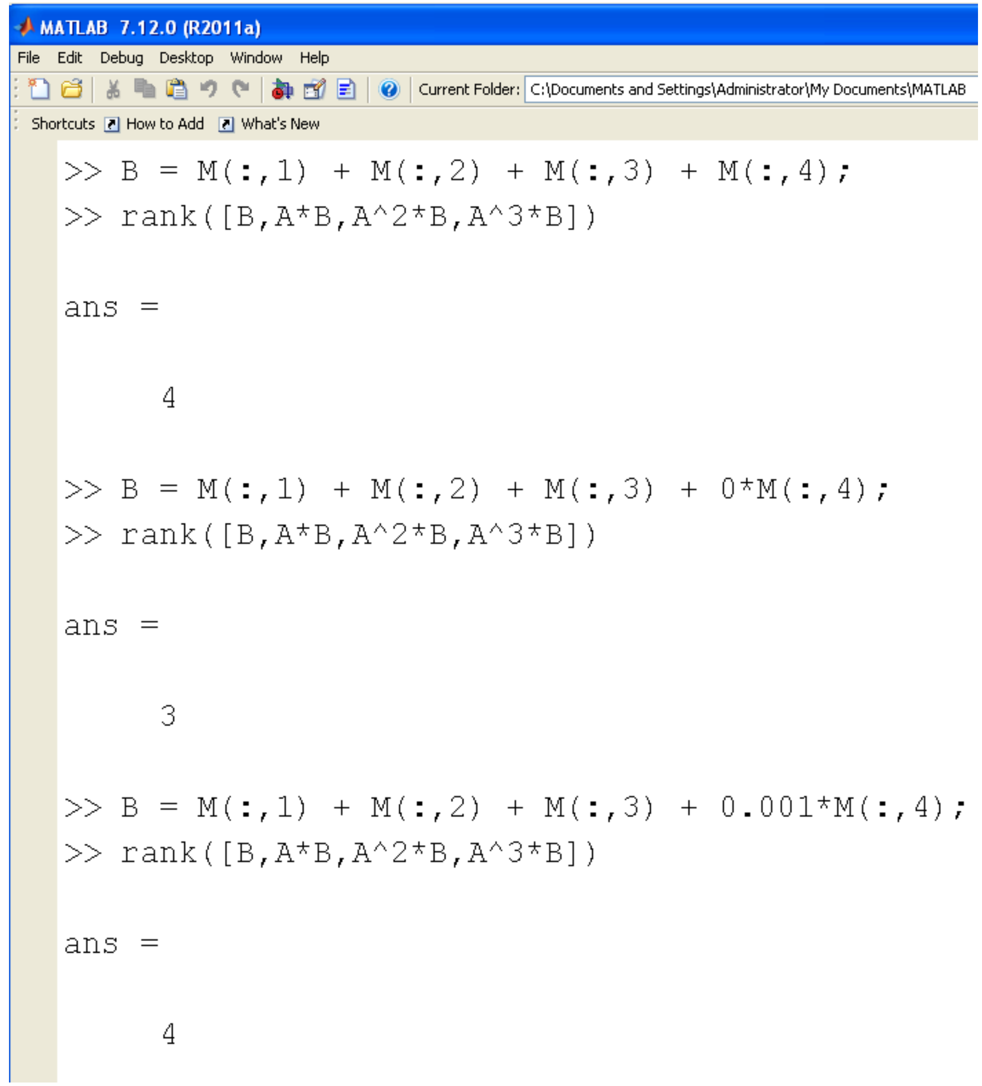
Rank & Eigenvectors:

If the B matrix includes all four eigenvetors, the system is controllable:

If B is missing one eigenvector, the system is not controllable.

This is a binary test:

- Even a small contribution of an eigenvector counts




```
>> B = M(:,1) + M(:,2) + M(:,3) + M(:,4);
>> rank([B,A*B,A^2*B,A^3*B])

ans =

     4

>> B = M(:,1) + M(:,2) + M(:,3) + 0*M(:,4);
>> rank([B,A*B,A^2*B,A^3*B])

ans =

     3

>> B = M(:,1) + M(:,2) + M(:,3) + 0.001*M(:,4);
>> rank([B,A*B,A^2*B,A^3*B])

ans =

     4
```

Determinant and Eigenvalues

- If all eigenvalues are non-zero, the matrix is full rank (the system is controllable)
- If an eigenvalue is *close* to zero, the system is weakly controllable / observable.

The determinant is the product of the eigenvalues

- If the determinant is not zero, the matrix is full rank (the system is controllable)
- If the determinant is *close* to zero, the system is weakly controllable / observable.

An eigenvalue or determinant close to zero tells you that one or modes is very difficult to control from the input.

Sidelight: The trace is the sum of the eigenvalues

- Interesting, but doesn't tell you much.

Example: Consider the 4-stage RC filter

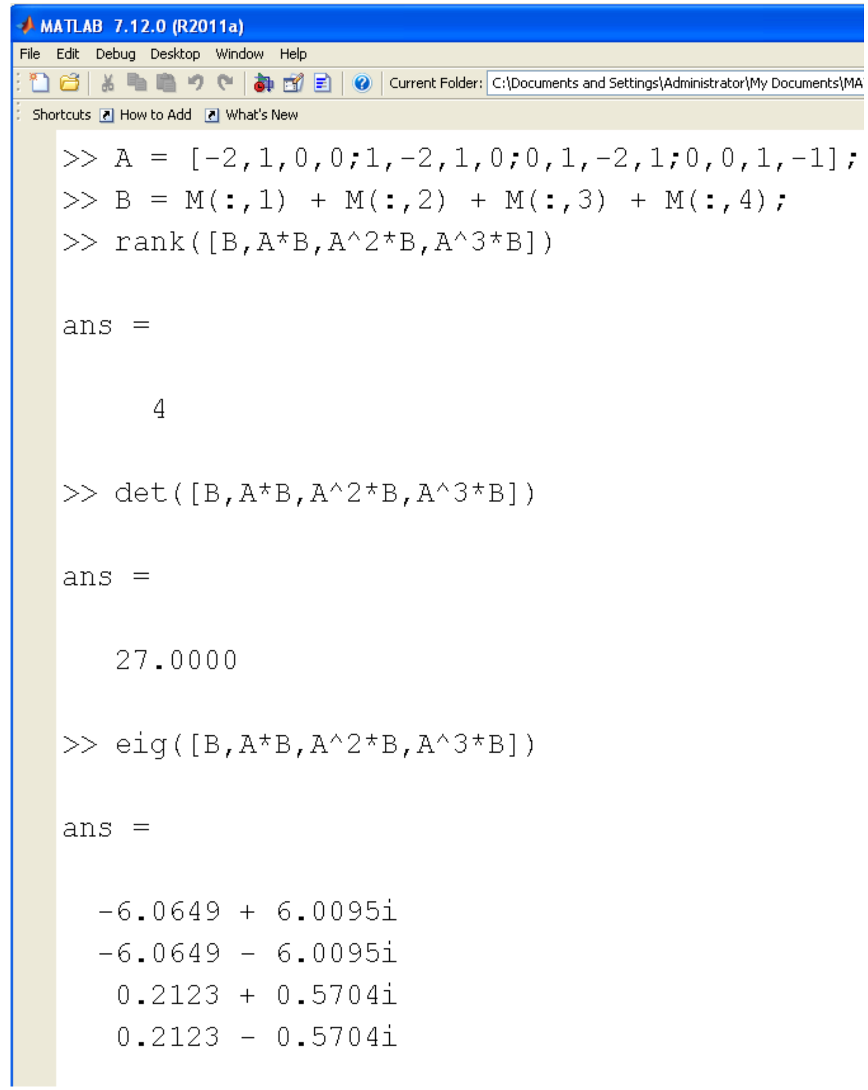
B includes all four eigenvectors:

The system is controllable

- The rank of the controllability matrix is 4

The system is strongly controllable

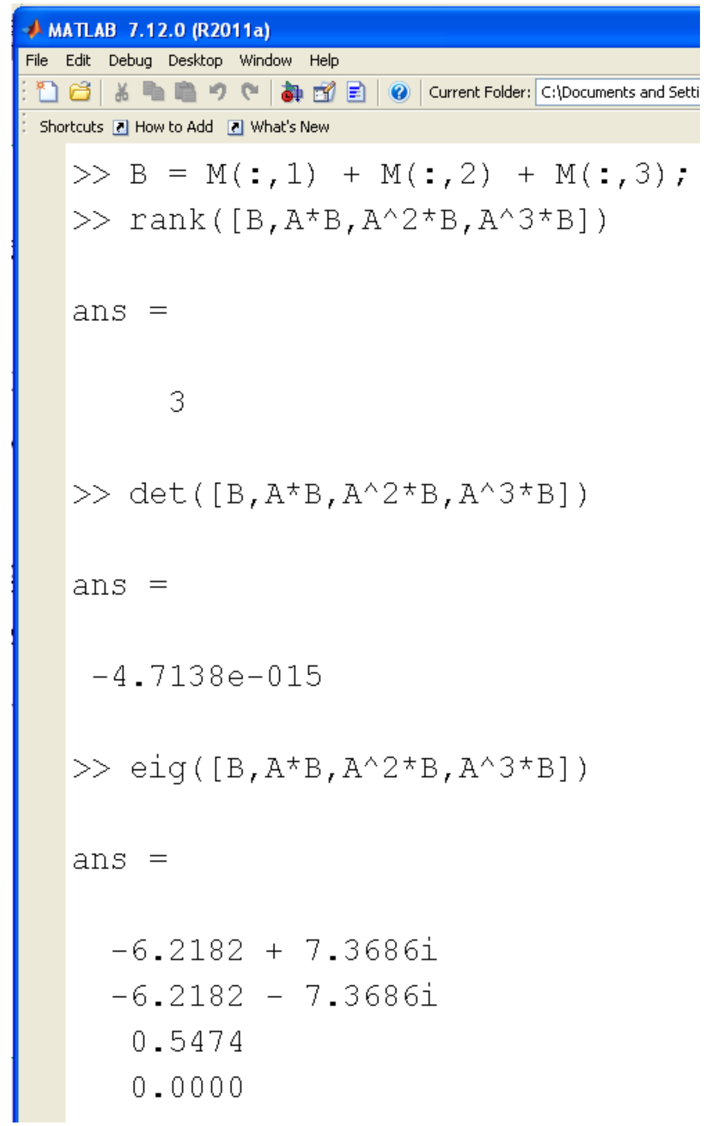
- The determinent is far from zero
- All four eigenvalues are far from zero



B contains only 3 eigenvectors

The system is uncontrollable

- The rank of the controllability matrix is 3
- The determinant is zero
- One eigenvalue is zero



B contains 4 eigenvectors

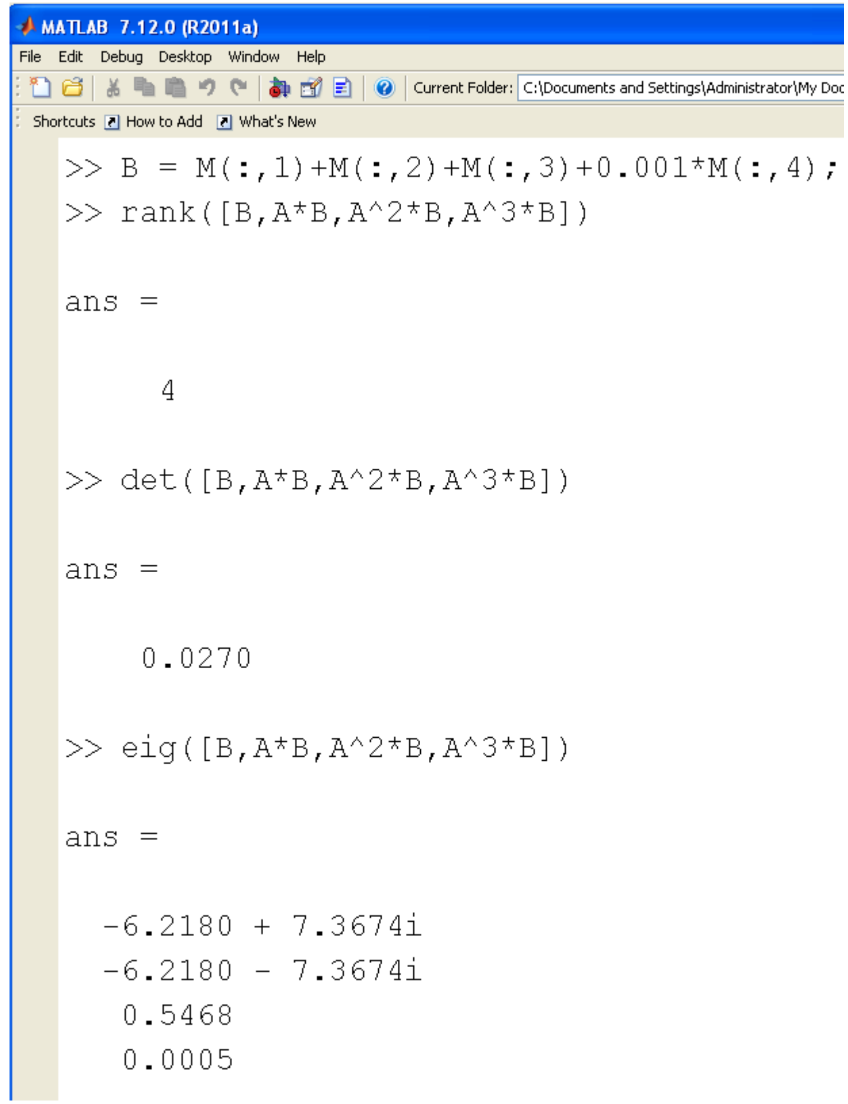
- One is very small
- The system is weakly controllable

The system is controllable

- The rank is 4

The system is weakly controllable

- The determinant is close to zero
- One eigenvalue is close to zero



PBH Test

A system is controllable if

$$\rho[A-\lambda I, B] = N$$

for all λ.

A system is observable if

$$\rho\left(\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}\right) = N$$

for all λ.

The PBH Rank Test tells you

- which modes are controllable / observable
- Which modes are not

Note: $[A - \lambda I]$ is full rank everywhere *except* when λ is an eigenvalue of A.

- You only need to check at the eigenvalues.
- If the rank is *not* N, then that mode is uncontrollable / unobservable.

Example: 4-stage RC filter where B only includes 3 eigenvectors:

- The 4th eigenvalue is uncontrollable

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1];
B = M(:,1) + M(:,2) + M(:,3);
P = eig(A);

rank([A - P(1)*eye(4,4), B])
ans =     4

rank([A - P(2)*eye(4,4), B])
ans =     4

rank([A - P(3)*eye(4,4), B])
ans =     4

rank([A - P(4)*eye(4,4), B])
ans =     3
```

This also works with the C matrix and observabilty:

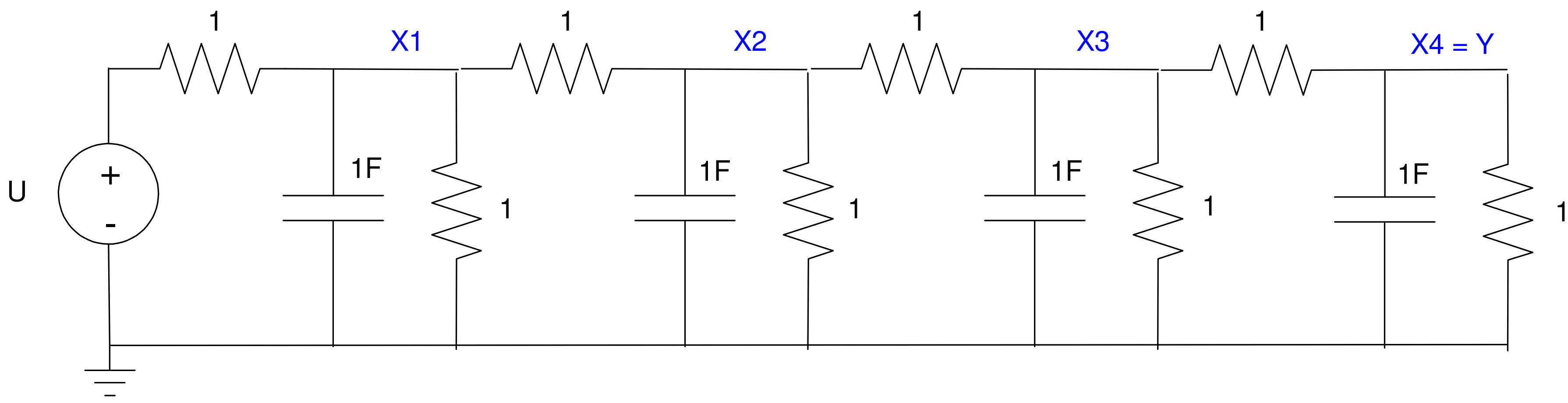
```
Mi = inv(M);
C = Mi(1,:) + Mi(2,:) + Mi(3,:);

rank([ C ; A - P(1)*eye(4,4) ] )
ans =     4

rank([ C ; A - P(2)*eye(4,4) ] )
ans =     4

rank([ C ; A - P(3)*eye(4,4) ] )
ans =     4

rank([ C ; A - P(4)*eye(4,4) ] )
ans =     3
```

The 4th eigenvalue is not observable.

Example 1: RC Filter

Is the following system controllable and observable?

- heat equation with four states



4-Stage RC filter

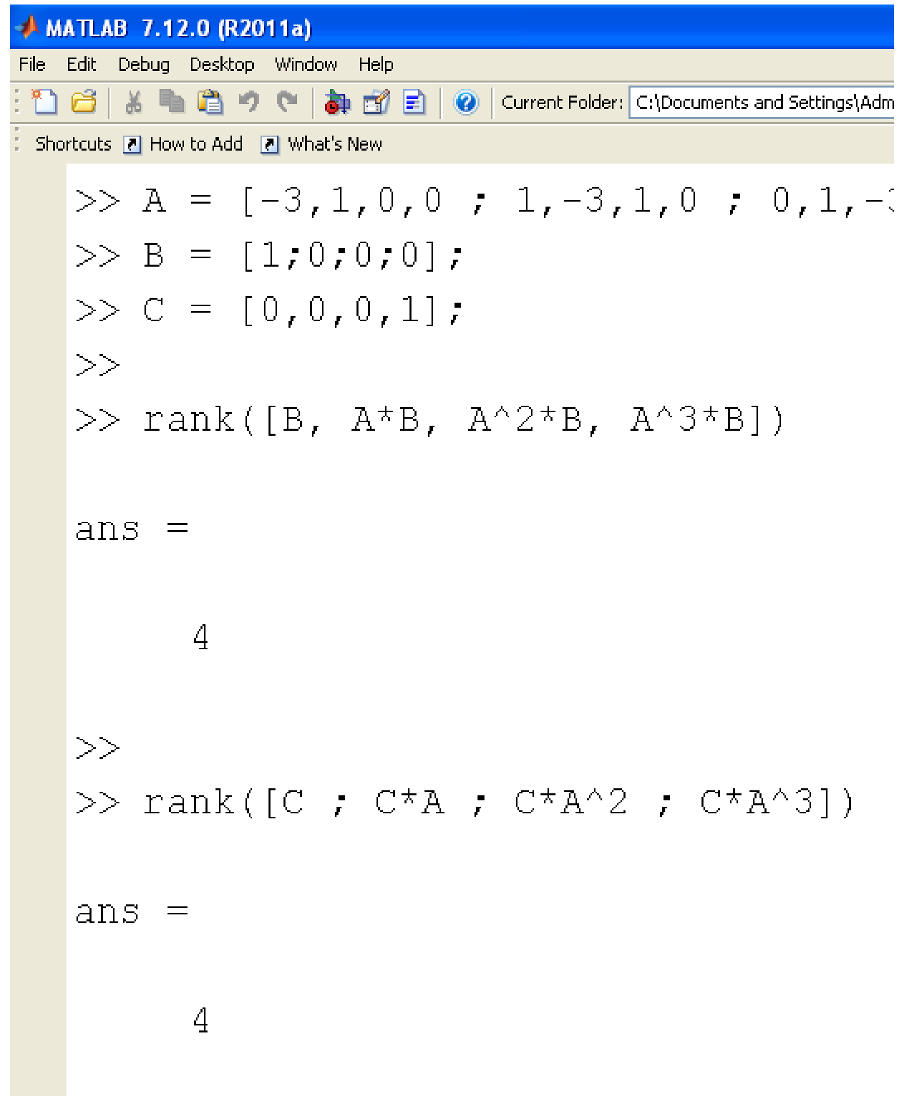
The dynamics are

$$sX = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

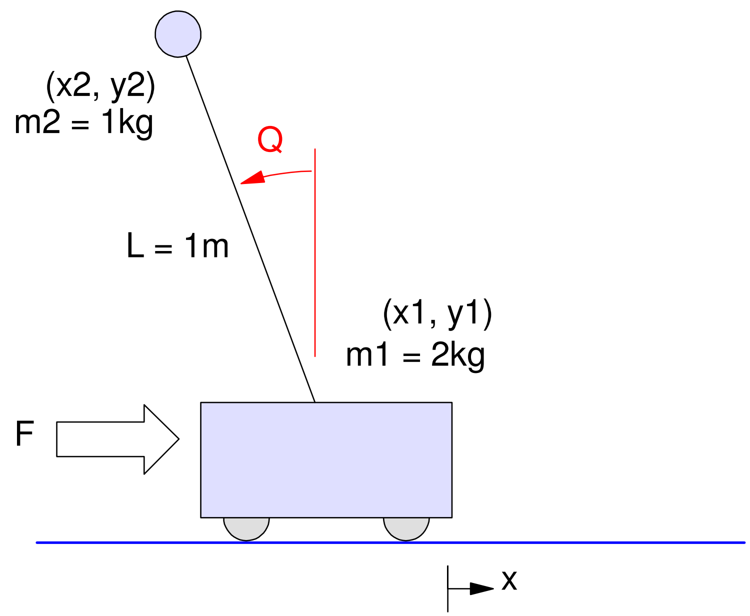
The controlability and observability matricies are full rank

- The system is controllable
- The sytem is observable



Example 2: Is a cart and pendulum controllable and observable?

$$s\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9 & 0 & 0 \\ 0 & 14.7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}F + \begin{bmatrix} 0 \\ 0 \\ -0.5 \\ 1.5 \end{bmatrix}T$$



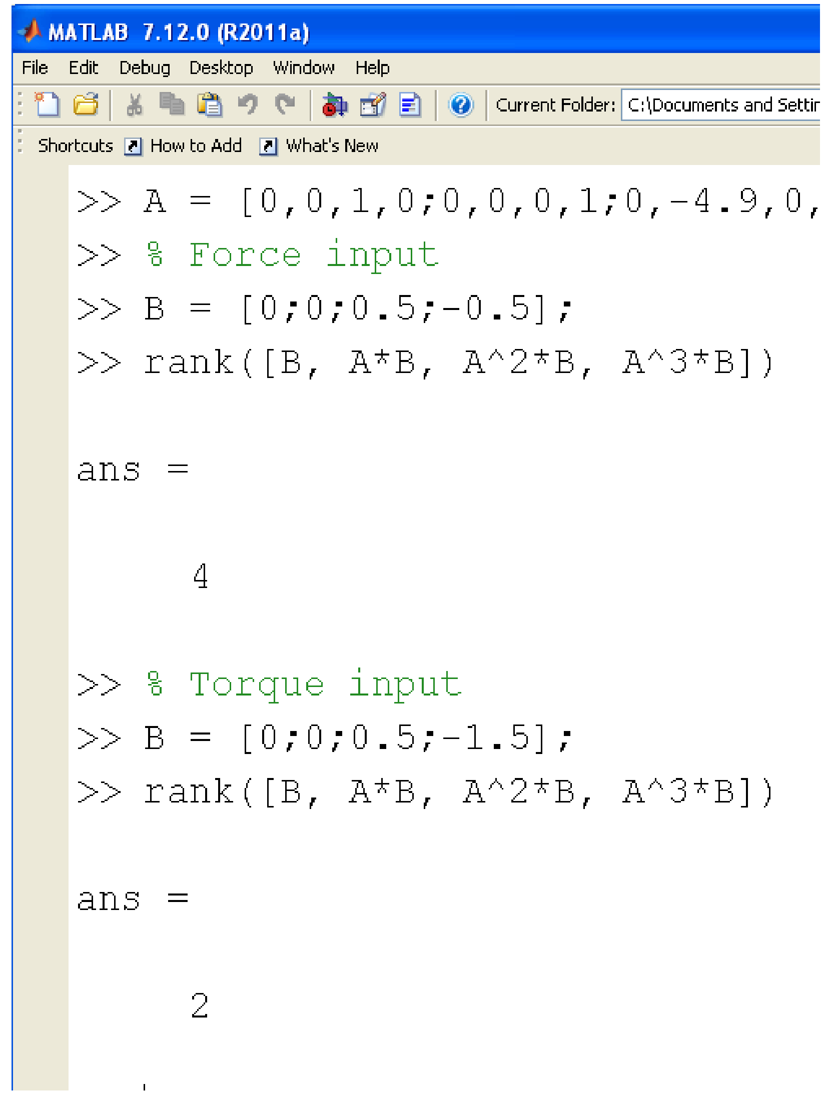
The system is controllable from F

- It is possible to balance a yardstick on your hand

The sustem is *not* controllable from T

- Something can't be controlled
- What?

Use the PBH test to determine what can't be controlled.

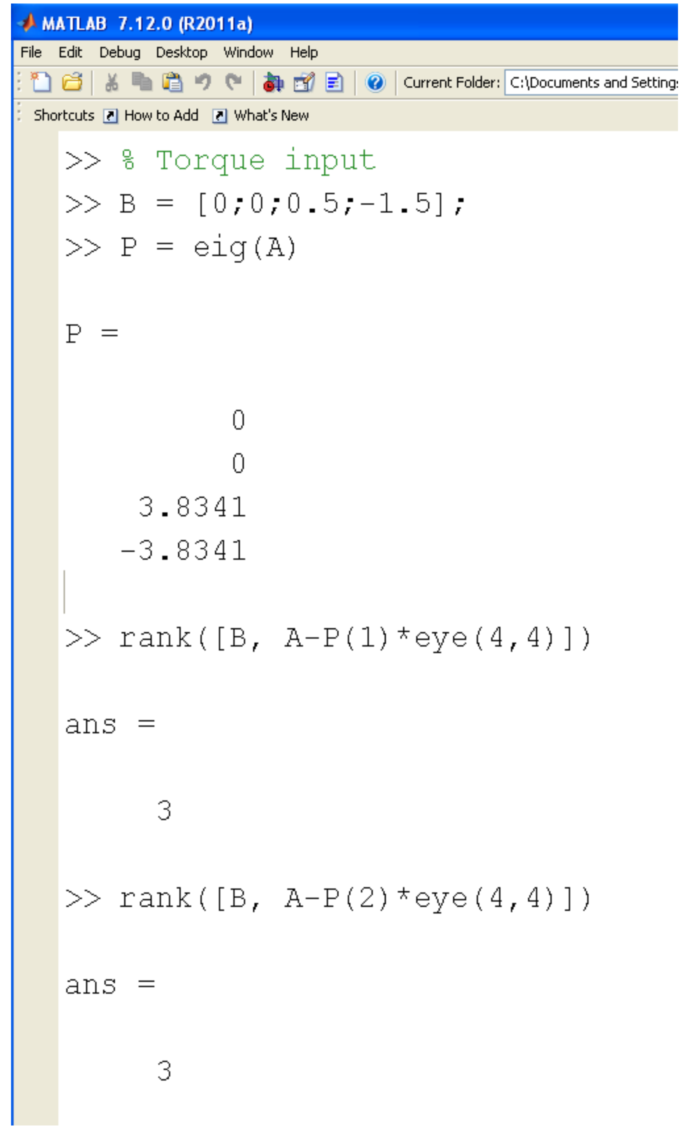


PBH Test

The first and second modes are not controllable

- Eigenvalue = 0
- Eigenvector = {position, velocity}

You can't control the cart's position or speed just by applying a torque on the beam



Observability

The system is observable from position (x)

- You can determine all four states just by looking at position

The system is strongly obsevable from position

- All eigenvalues of the observability matrix are far from zero
- The determinenat far from zero

Surprising, but that's what the math tells you

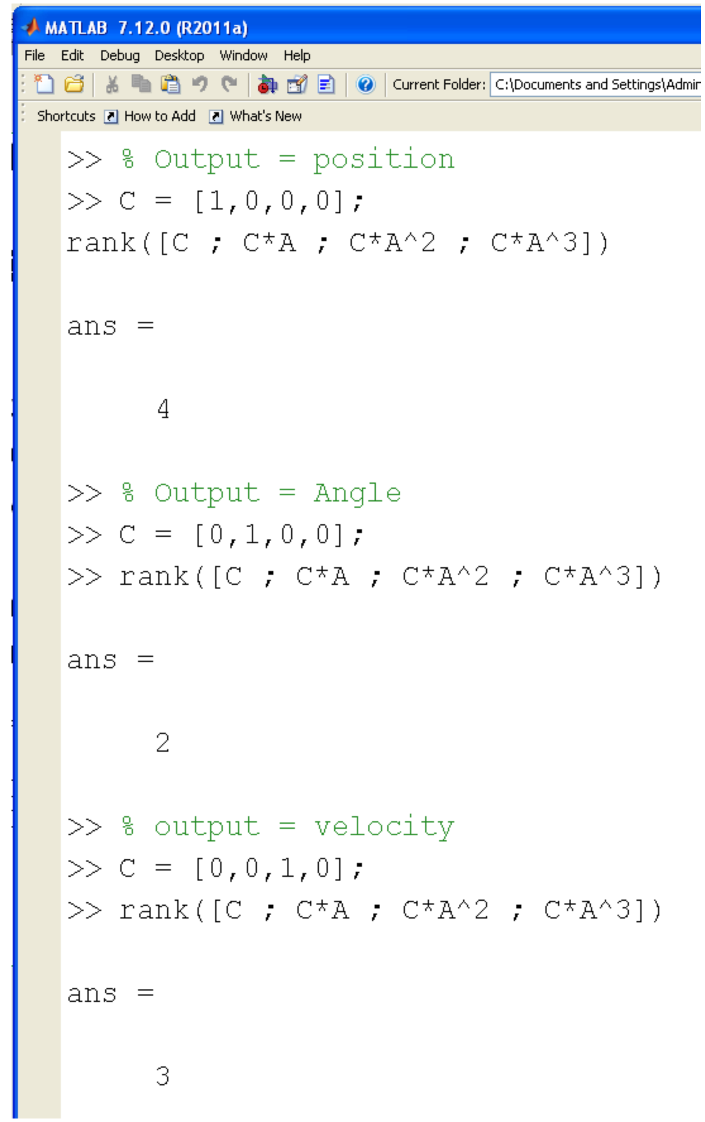
```
>> % Output = Position
>> C = [1,0,0,0];
>> rank([C ; C*A ; C*A^2 ; C*A^3])

ans =

     4

>> eig([C ; C*A ; C*A^2 ; C*A^3])

ans =

        0 + 2.2136i
        0 - 2.2136i
   1.0000
  -4.9000

>> det([C ; C*A ; C*A^2 ; C*A^3])

ans =

  -24.0100
```

The system is not observable from angle

- Rank = 2
- Two modes can't be seen

The system is not observable from velocity

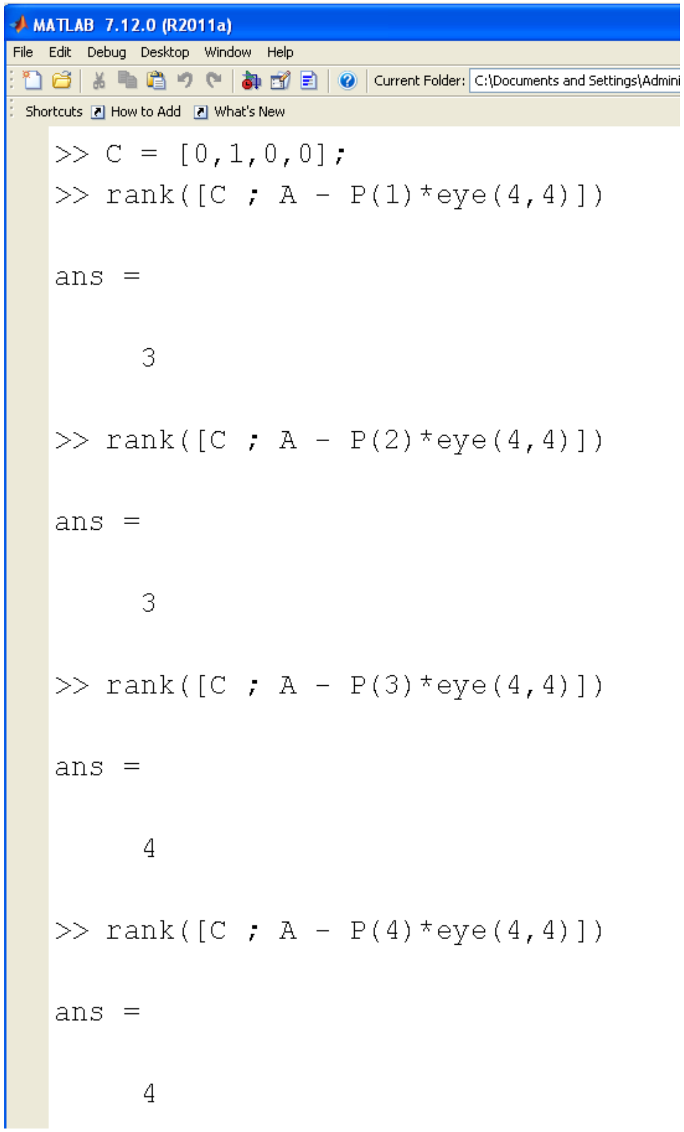
- Rank = 3
- One mode can't be seen



PBH Rank test tells you which modes are not observable

- Measure angle
- Position (first eigenvector) is not observable
- Velocity (second eigenvector) is not observable

```
rank([C ; A – P(1)*eye(4,4)])
   ans =      3

rank([C ; A – P(2)*eye(4,4)])
   ans =      3

rank([C ; A – P(3)*eye(4,4)])
   ans =      4

rank([C ; A – P(4)*eye(4,4)])
   ans =      4
```



Summary

A system is *controllable* if the input can drive you to an arbitrary state in fininte time

A system is *observable* if you can determine the states using only the output and its derivatives

Several tests can be used to determine if a system is controllable or observable:

- Rank of Controllability / Observability Matrix
- PBH test
- B & C matrix contains all eigenvectors
- B & C matrix has non-zero entries when expressed in Jordan form