Reduced-Order Observers

NDSU ECE 463/663

Lecture #22

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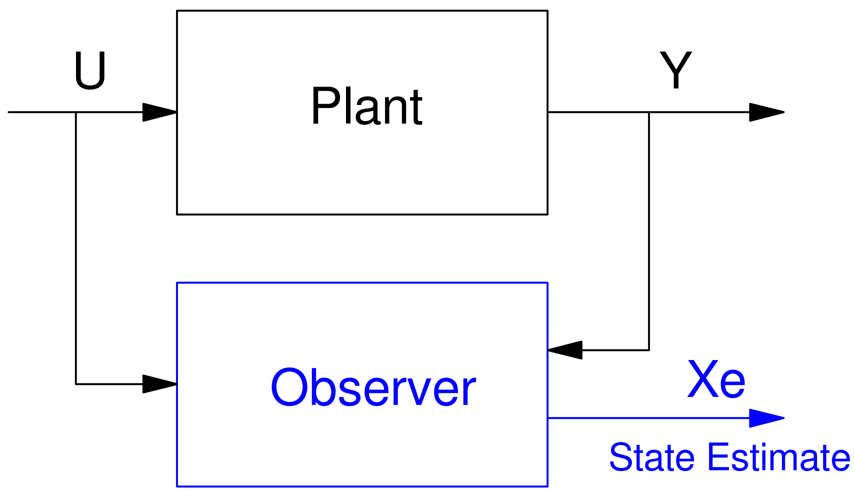
Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Full-Order Observers (recap)

- Full-state feedback assumes all states are known
- A full order observer estimates states that are not measured

This creates a system of order 2N

- N states for the plant
- Another N states for the observer



Reduced-Order Observer

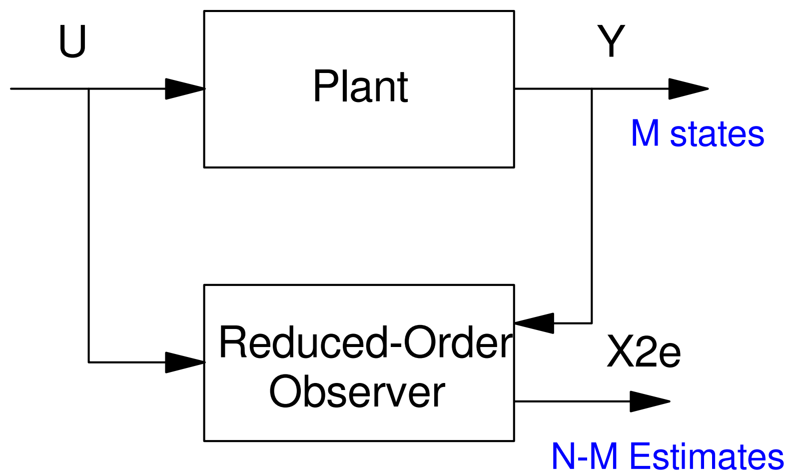
If M states are measured directly, they do not need to be estimated

Can you only estimate the remain N-M states?

- Creating a lower-order system?

Answer: yes

- Result is a reduced-order observer



Derivation

Luenberger D G. Observers for multivariable systems. IEEE Trans. Automat. Contr. AC-11:190-7, 1966.

Assume you have a dynamic system which is observable:

$$sX = AX + BU$$

$$Y = CX$$

Assume that some of the states are measured directly. Separate X into the states which are directly measured and those which are not:

$$s\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}U$$

$$Y = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where C_1 is inevitable (i.e. X1 is measured directly).

A full-order observer is

$$s\begin{bmatrix} X_{1e} \\ X_{2e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} X_{1e} \\ X_{2e} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}U + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}(Y - C_1 X_e)$$

Since X1 is measured directly, let

$$X_{1e} = C_1^{-1} Y$$

Then, the full-order observer becomes

$$sX_{2e} = A_{21} C_1^{-1} Y + A_{22} X_{e2} + B_2 U$$

Let

$$X_{2e} = LY + Z$$

where Z is from the system

$$sZ = FZ + GY + HU$$

Then

$$sE_2 = sX_2 - sX_{e2} = A_{21}X_1 + A_{22}X_2 + B_2U - LsY - sX$$

$$sE_2 = A_{21}X_1 + A_{22}X_2 + B_2U - L(C_1(A_{11}X_1 + A_{12}X_2 + B_1U))$$

but

$$Z = X_{e2} - LY = X_2 - E_2 - LY = X_2 - E_2 - LC_1X_1$$

so

$$\begin{aligned} sE_2 &= A_{21}X_1 + A_{22}X_2 + B_2U - L(C_1(A_{11}X_1 + A_{12}X_2 + B_1U)) \\ &\quad -F(X_2 - E_2 - LC_1X_1) - GY - HU \end{aligned}$$

Grouping terms

$$sE_2 = FE_2 + (A_{21} - LC_1A_{11} - GC_1 + FLC_1)X_1$$

$$+(A_{22} - LC_1A_{12} - F)X_2$$

$$+(B_2 - LC_1B_1 - H)U$$

In order for the error to be driven to zero,

$$GC_1 = A_{21} - LC_1A_{11} + FLC_1$$

$$F = A_{22} - LC_1A_{12}$$

$$H = B_2 - LC_1B_1$$

Then, the error dynamics become

$$sE_2 = FE_2$$

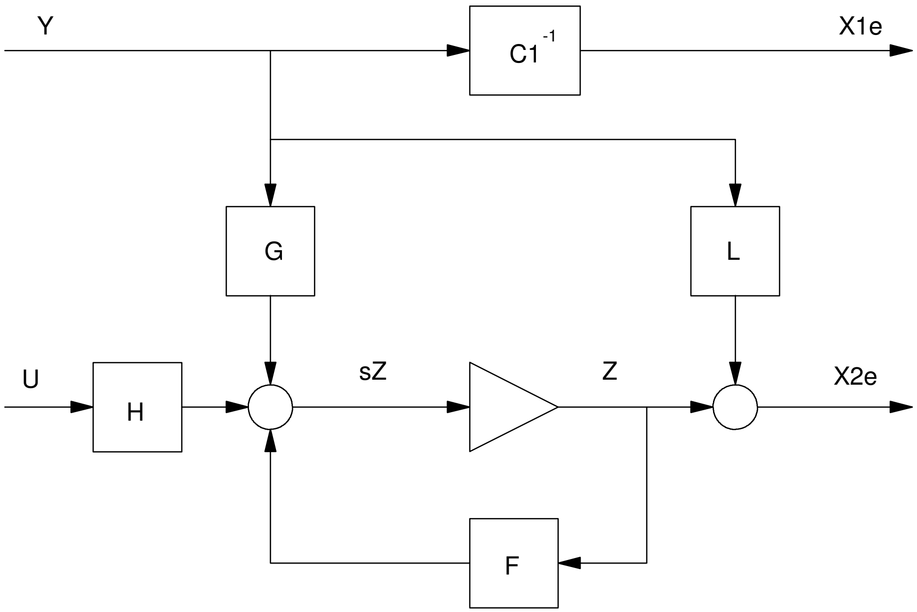
If F is chosen so that it is negative definite, the error will be driven to zero. To do this, you can use pole-placement to find L to place the poles of:

$$F = A_{22} - L(C_1 A_{12})$$

The augmented system is then

$$s\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ H \end{bmatrix} U$$

$$\begin{bmatrix} X_{1e} \\ X_{2e} \end{bmatrix} = \begin{bmatrix} C_1^{-1}C & 0 \\ LC & I \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}$$



Example:

Reduced-Order Observer for a heat equation. Let

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

Design a reduced-order observer to estimate states 1..3

Rewrite this as

$$s\begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = \begin{bmatrix} -1 & \vdots & 1 & 0 & 0 \\ \cdots & & \cdots & \cdots & \cdots \\ 1 & \vdots & -2 & 1 & 0 \\ 0 & \vdots & 1 & -2 & 1 \\ 0 & \vdots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

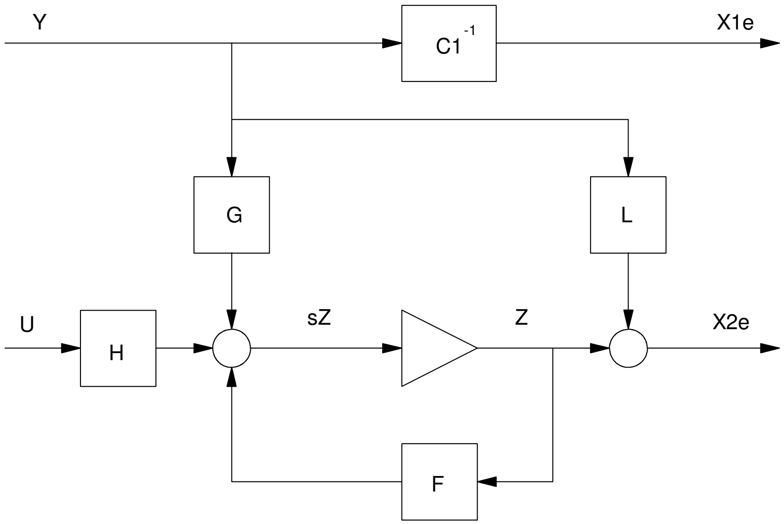
$$Y = \begin{bmatrix} 1 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix}$$

Pick L to stabilize F:

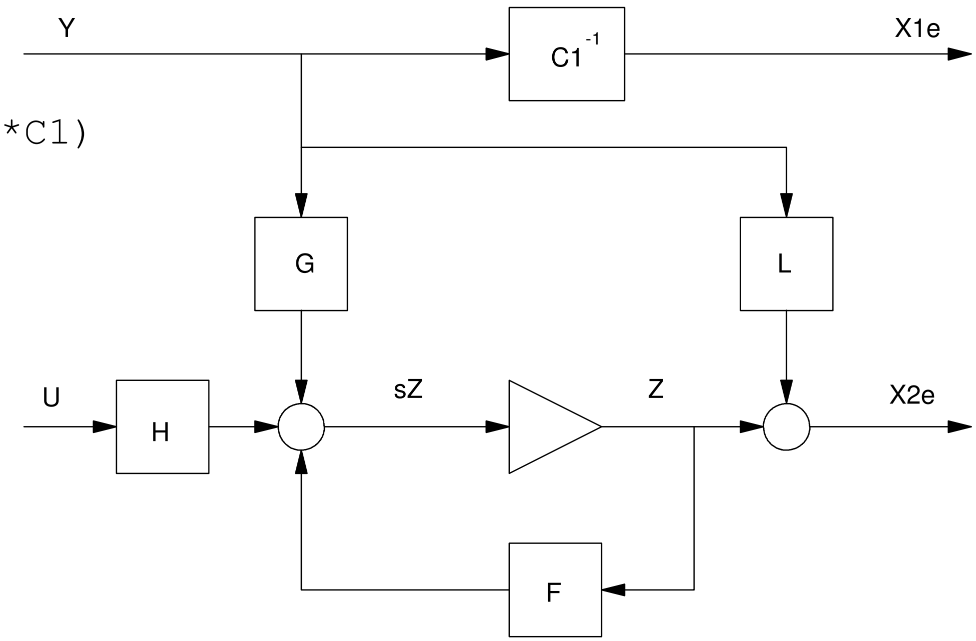
$$F = A_{22} - L(C_1 A_{12})$$

In Matlab

```
A11 = -1;
A12 = [1,0,0];
A21 = [1;0;0];
A22 = [-2,1,0;1,-2,1;0,1,-2];

B1 = 0;
B2 = [0;0;1];

C1 = 1;
C2 = [0,0,0];

L = ppl(A22',(C1*A12)',[-3,-4,-5])'

    6.
    13.
    12.


F = A22 - L*C1*A12

  - 8.      1.     0.
  - 12.   - 2.     1.
  - 12.     1.   - 2.
```



```
H = B2-L*C1*B1

     0.
     0.
     1.

G = inv(C1) * (A21 - L*C1*A11 + F*L*C1)

  - 28.
  - 73.
  - 71.

eig(F)

  - 5.
  - 4.
  - 3.
```



The system plus the reduced-order observer is then:

```
A7 = [A, zeros(4,3);G*C,F]

  - 1.     1.     0.     0. :   0.      0.     0.
    1.   - 2.     1.     0. :   0.      0.     0.
    0.     1.   - 2.     1. :   0.      0.     0.
    0.     0.     1.   - 2. :   0.      0.     0.
  - - - - - - - - - - - - :- - - - - - - - - - -
  - 28.    0.     0.     0. : - 8.      1.     0.
  - 73.    0.     0.     0. : - 12.   - 2.     1.
  - 71.    0.     0.     0. : - 12.     1.   - 2.


B7 = [B;H]

    0.
    0.
    0.
    1.
- - - -
    0.
    0.
    1.
```

```
Cx = [C,zeros(1,3);L*C,eye(3,3)]

     1.     0.     0.     0.     0.     0.     0.
     6.     0.     0.     0.     1.     0.     0.
     13.    0.     0.     0.     0.     1.     0.
     12.    0.     0.     0.     0.     0.     1.
```

Plotting the response of the observer with the plant having an initial condition of {1, 2, 3, 4} is:

First, create the augmented system (plant plus reduced order observer)

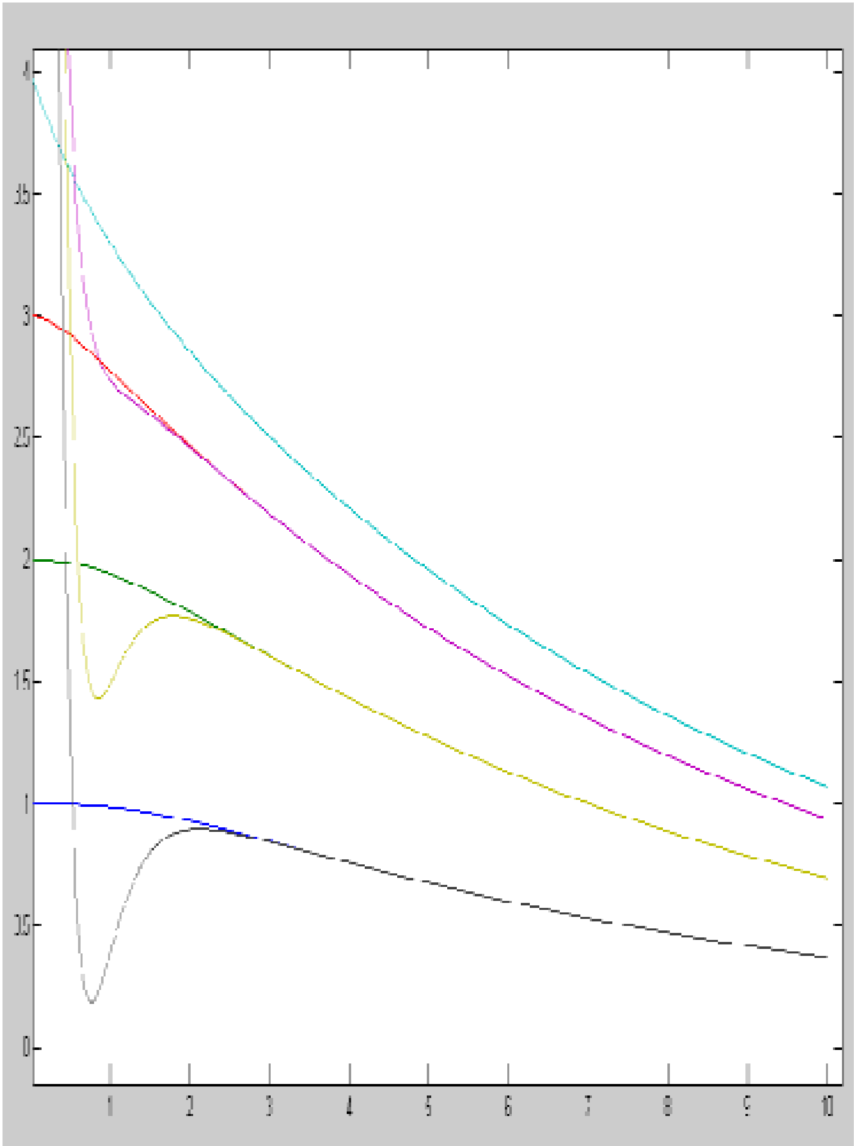
```
A7 = [A, zeros(4,3);G*C,F]
B7 = [B;H]
Cx = [C,zeros(1,3);L*C,eye(3,3)]
```

To see if the observer states converge, plot all four plant states as well:

```
C7 = [eye(3,4), zeros(3,3);  Cx]

    1.0000         0         0         0         0         0         0
         0    1.0000         0         0         0         0         0
         0         0    1.0000         0         0         0         0
         0         0         0    1.0000         0         0         0
         0         0         0    6.0000    1.0000         0         0
         0         0         0   13.0000         0    1.0000         0
         0         0         0   12.0000         0         0    1.0000

>> X0 = [1;2;3;4;0;0;0]

     1    plant states
     2
     3
     4
   - - -
     0
     0        reduced-order observer states
     0

>> G7 = ss(A7, X0, C7, zeros(7,1));
>> y7 = impulse(G7,t);
>> plot(t,y7)
```

Results

- The observer states converge
- The settling time is about 2 seconds (observer poles at -3, -4, -5)
- You get fewer states, which isn't that big of a deal since it will be implemented in software
- In exchange for a much more complicated design - which is a big deal.



Summary

Reduced-Order Observers *do* exist

However

- They are *much* more complicated
- They are more sensitive to modeling errors
- They are more sensitive to noise

It is usually better to use a full-order observer

Reduced-order observers are covered for completeness