Calculus of Variations

NDSU ECE 463/663

Lecture #23

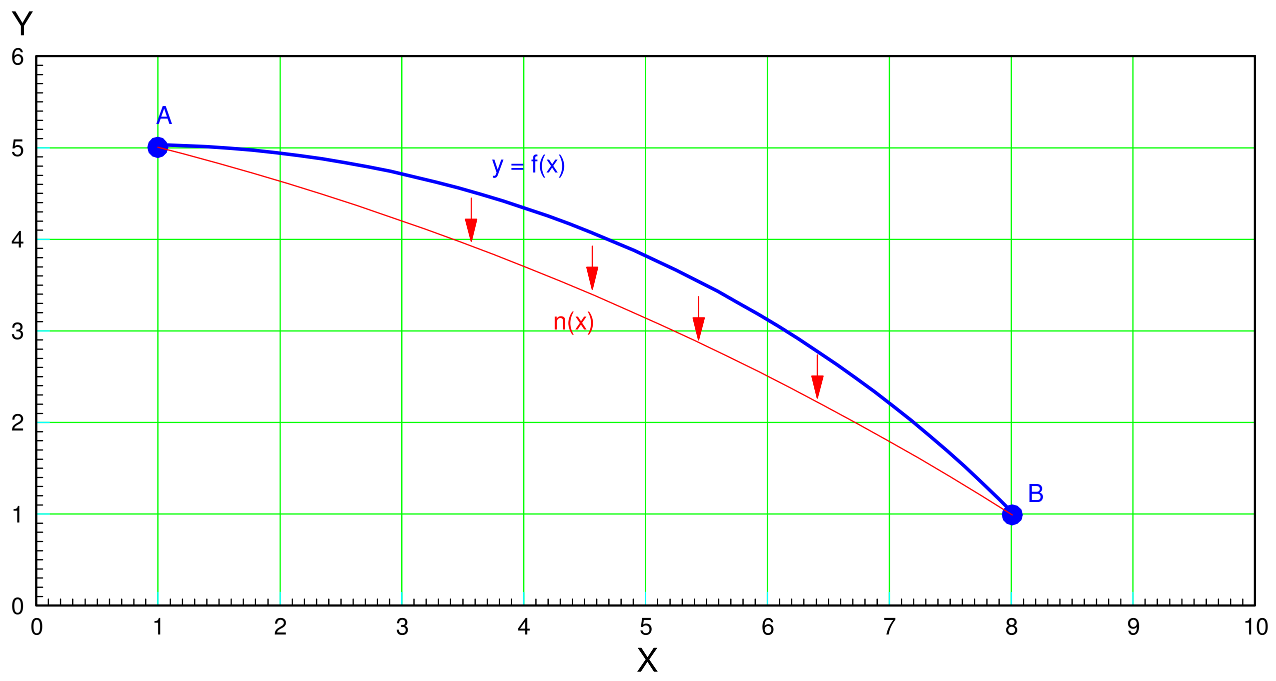
Inst: Jake Glower

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Calculus of Variations

Calculus of Variations is a branch of mathematics dealing with optimizing functionals. A functional is a function of functions. For example

$$J(x) = \int_a^b F(t, x, \dot{x})dt$$



Find the function, f(x), which minimizes a functional

Euler Legrange Equation

The minimum is found from

$$\frac{dJ}{d\xi} = \lim_{\xi \to 0} \left(\frac{J(x+\xi n) - J(x)}{\xi} \right) = 0$$

The solution is (see lecture notes for derivation)

$$\frac{dJ}{d\xi} = \int_a^b \left(F_x + \frac{d}{dt}(F_{\dot{x}}) \right) n \cdot dt + (F_{\dot{x}} n)\Big|_a^b = 0$$

Since n(t) is an arbitrary function, this can only be true if

$$\boxed{F_x - \frac{d}{dt}(F_{\dot{x}}) = 0}$$

Euler LeGrange Equation

Boundary Conditions

$$\frac{dJ}{d\xi} = \int_a^b \left(F_x + \frac{d}{dt}(F_{\dot{x}}) \right) n \cdot dt + (F_{\dot{x}} n)\big|_a^b = 0$$

Fixed Endpoints:

x(t) is defined by the endpoint constraints

Free Endpoints:

$$F_{\dot{x}} = 0$$

Example 1: Shortest Distance Between Two Points:

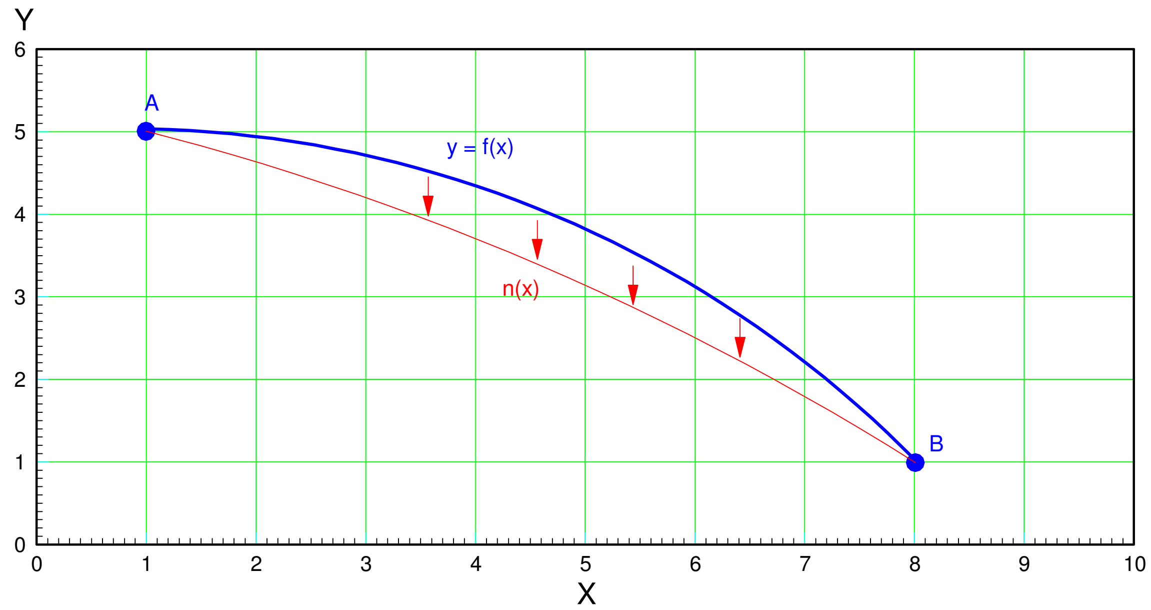
What function minimizes the distance traveled from point A to point B?

- What is the minimal cost road connecting these two points assuming uniform cost per mile

J = length:

$$J = \int_a^b \left(\sqrt{dx^2 + dy^2} \right)$$

$$J = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx$$



Functional:

$$F = \sqrt{1 + (y')^2}$$

The Euler Legrange equation is

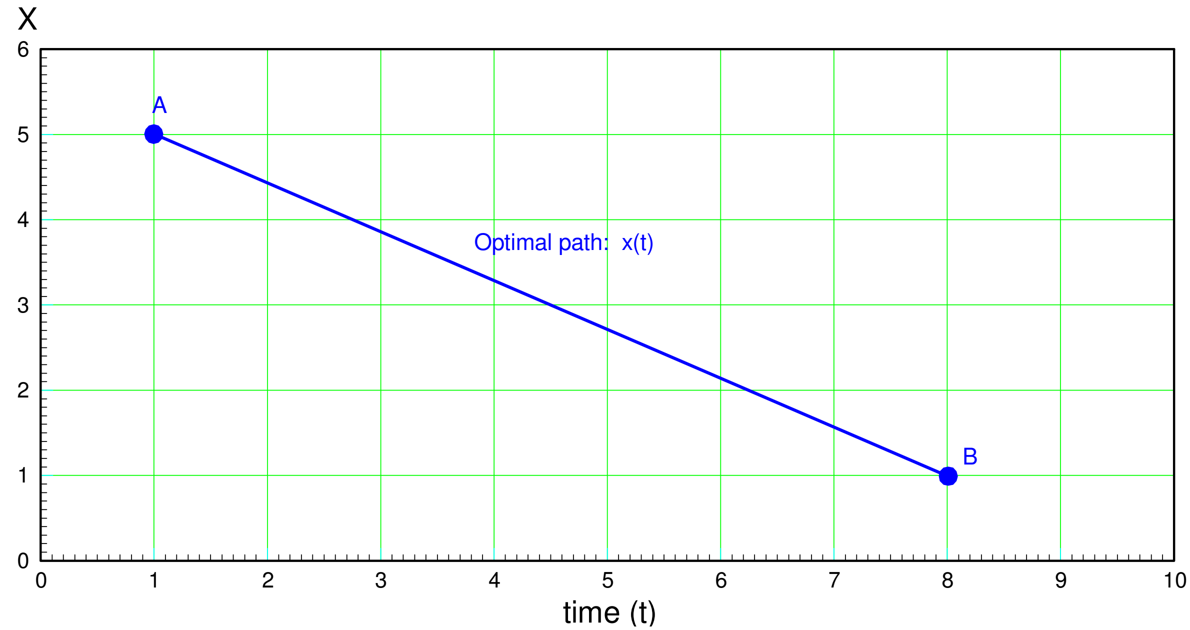
$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{y'}{\sqrt{1+y'^2}} = c$$

$$y' = a$$

$$y = ax + b$$



A straight line is the shortest distance between two points

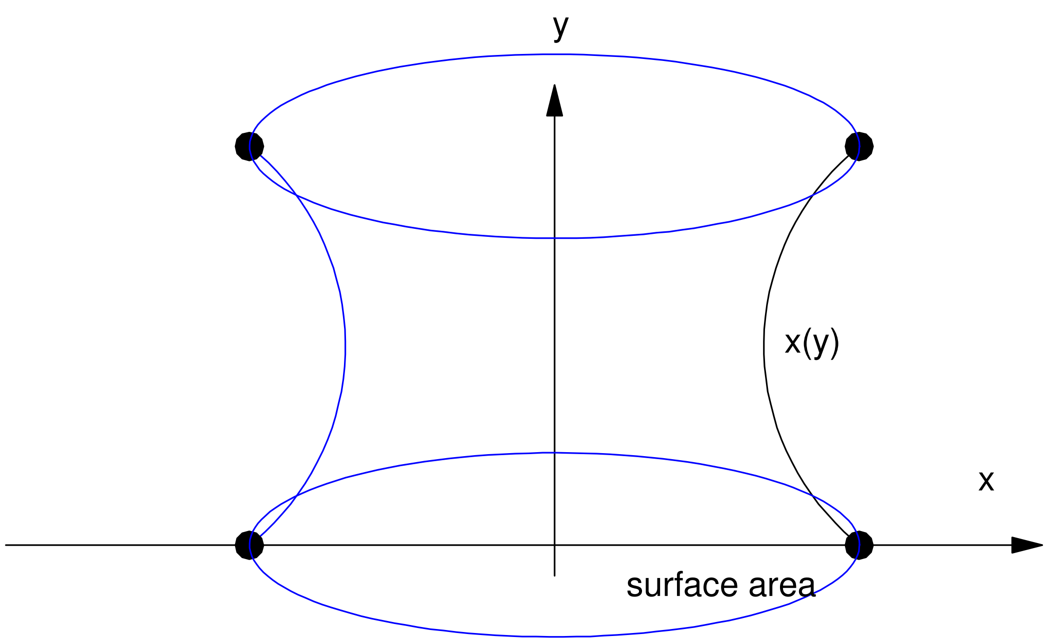
Example 2: Soap Film:

Minimize the surface area of a soap film:

- What road connects the points (0, 5) and (5,5) assuming the cost per mile is proportional to X?

Functional:

$$F = x\sqrt{1 + y'^{\,2}}$$



Soution: The function must satisfy the Euler LeGrange equation:

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

Since there is no 'y', this simplifies to

$$-\frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^{\,2}}}\right) = 0$$

$$\frac{xy'}{\sqrt{1+y'^{\,2}}} = a$$

Solve for y'

$$y' = \frac{a}{\sqrt{x^2-a^2}}$$

$$dy = \frac{a}{\sqrt{x^2-a^2}}\,dx$$

$$\int dy = \int \left(\frac{a}{\sqrt{x^2 - a^2}} \right) dx$$

Change of variable:

$$x = a \ \cosh(\theta)$$

$$dx = a \sinh(\theta) \cdot d\theta$$

Plugging in

$$\int dy = \int \left(\frac{1}{\sqrt{\cosh^2(\theta) - 1}} \cdot (a \sinh(\theta)) \right) \cdot d\theta$$

$$y = \int \left(\frac{a \sinh(\theta)}{\sinh(\theta)} \right) \cdot d\theta = a\theta + b$$

Resubstituting for x

$$y = a\left(\cosh^{-1}\left(\frac{x}{a}\right) \right) + b$$

$$\boxed{x = a\cosh\left(\frac{y-b}{a}\right)}$$ *shape of a soap film circling the y-axis*

$$\boxed{y = a\cosh\left(\frac{x-b}{a}\right)}$$ *shape of a soap film circling the x-axis*

Endpoints:

y(0) = 5 left endpoint

y(5) = 5 right endpoint

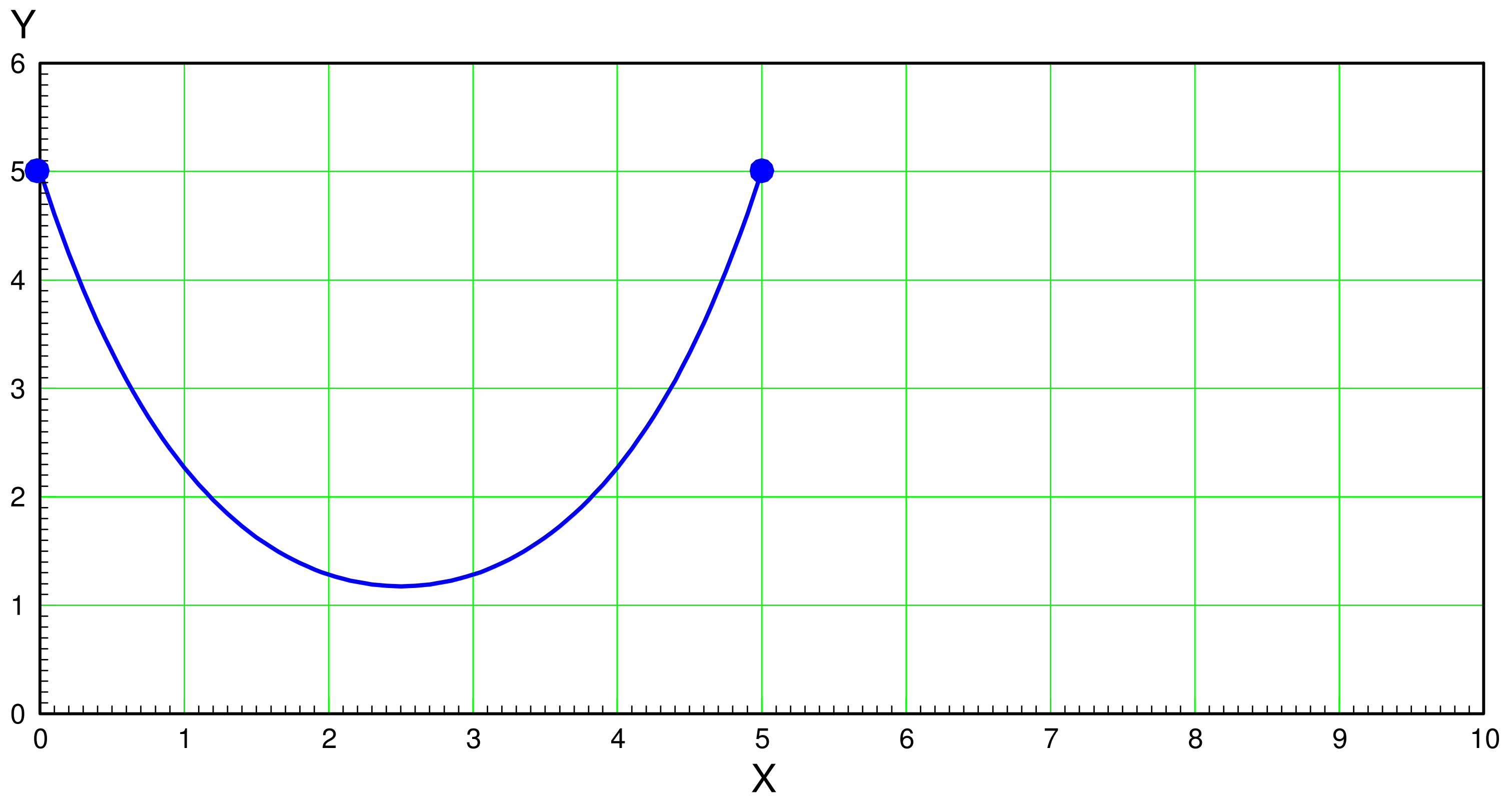
2-equations for 2 unknowns:

$$5 = a\cosh\left(\frac{0-b}{a}\right)$$

$$5 = a\cosh\left(\frac{5-b}{a}\right)$$

Solution: (matlab and fminsearch)

$$y = 1.175\cosh\left(\frac{x-2.5}{1.175}\right)$$



Soap Film with a Free Endpoint:

If the right endpoint is free, then the constraint is that

$$F_{y'} = 0$$

Example:

- Left Endpoint: x = 0, y = 5
- Right Endpoint: x = 2, y = free

Left Endpoint:

$$5 = a\cosh\left(\frac{-b}{a}\right)$$

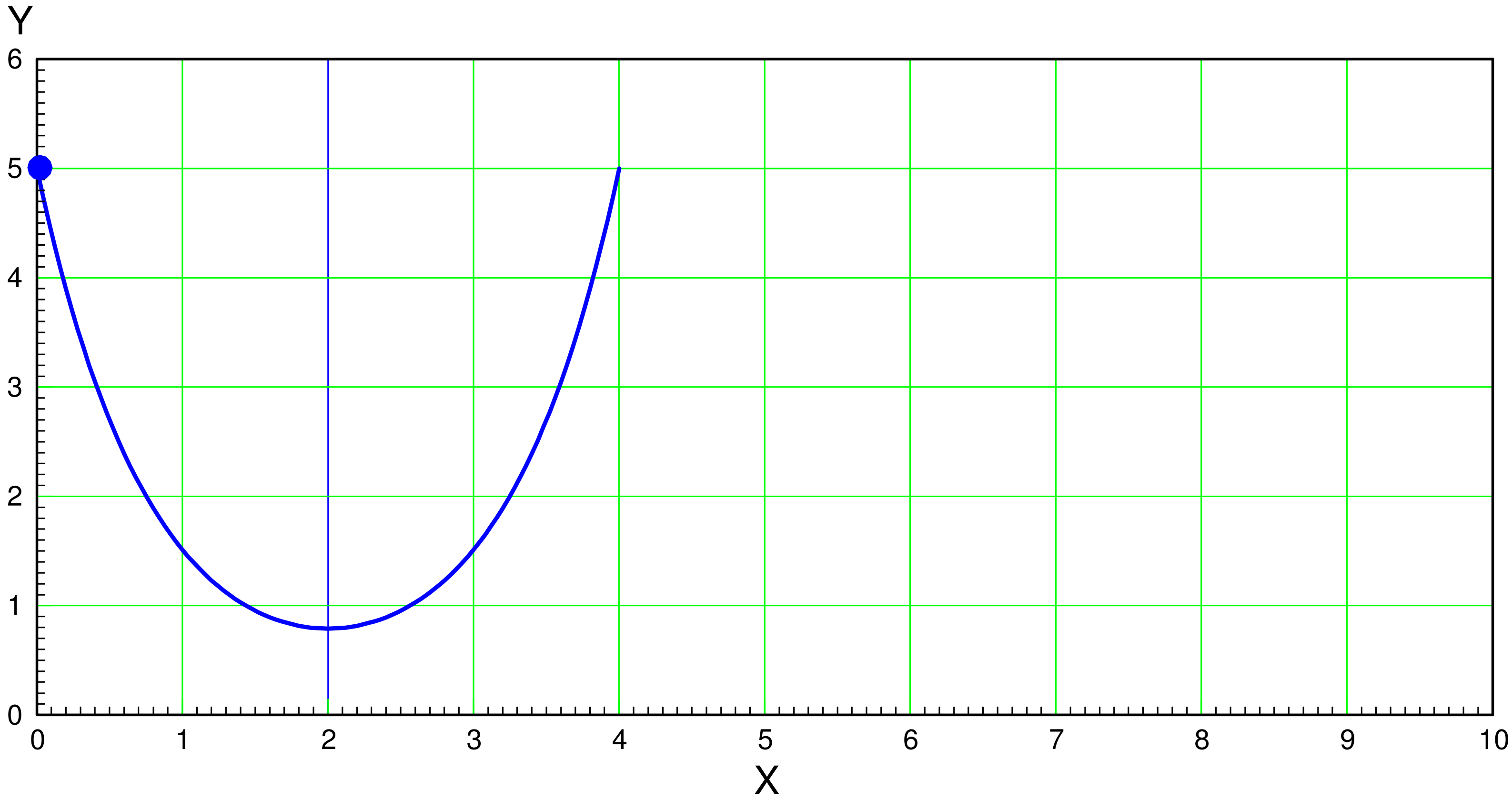
Right Endpoint:

$$F_{y'} = 0 \Rightarrow y' = 0$$

$$y' = -a\sinh\left(\frac{x-b}{a}\right) = 0$$

Solution:

$$y = 0.7898\cosh\left(\frac{x-2}{0.7898}\right)$$



Euler Legrange Equation with Two Dependent Variables

If

$$J = \int_a^b F(t, x, \dot{x}, u, \dot{u})dt$$

then

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$F_u - \frac{d}{dt}(F_{\dot{u}}) = 0$$

Euler Legrange Equation with Contraints:

If

$$G(t, x, \dot{x}, u, \dot{u}) = 0$$

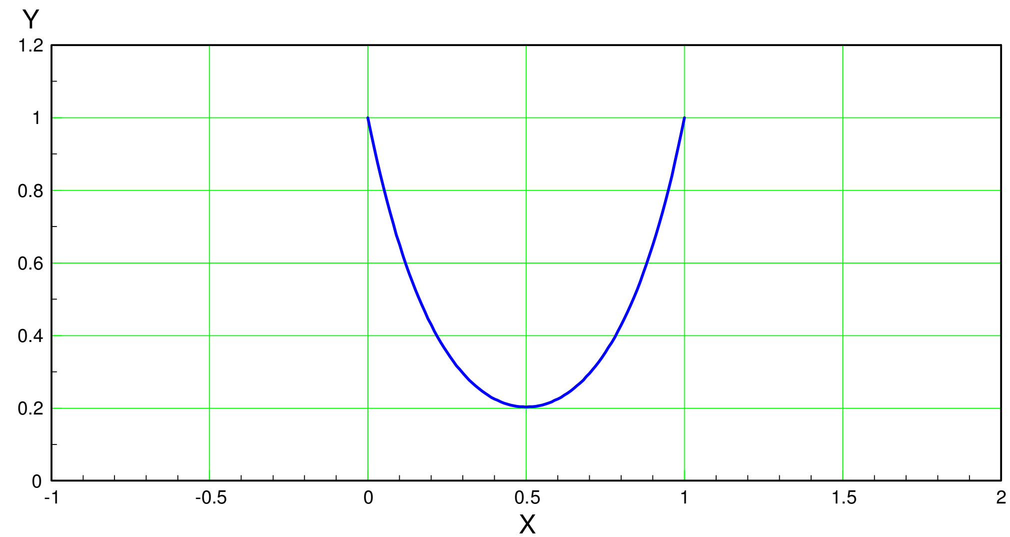
then add a Legrange multiplier, M:

$$J = \int_a^b (F(t, x, \dot{x}, u, \dot{u}) + M \cdot G(t, x, \dot{x}, u, \dot{u}))dt$$

Example 3: Hanging Chain

Find the shape of a hanging chain

- Endpoints 1m apart
- Length of chain = 2m



Functional: Minimize the potential energy:

$$F = x\sqrt{1 + y'^{\,2}}$$

Subject to

- y(0) = 1;
- y(1) = 1
- $length = \int_0^1 \sqrt{1 + y'^2}\,dx = 2$

Solution: Add a LaGrange multuiplier

$$F = x\sqrt{1+y'^{\,2}} + M\sqrt{1+y'^{\,2}}$$

Plugging into the Euler Legrange equation:

$$F_y - \tfrac{d}{dx}(F_{y'}) = 0$$

There is no y, so

$$F_{y'} = a$$

$$y' = \frac{a}{\sqrt{(x+M)^2 - a^2}}$$

$$\int dy = \int \frac{a}{\sqrt{(x+M)^2 - a^2}}dx$$

Do a change of variables

$$x + M = z$$

$$dx = dz$$

then

$$\int dy = \int \frac{a}{\sqrt{z^2 - a^2}} dx$$

Do another change in variables

$$z = a \cdot \cosh(w)$$

$$dz = a \cdot \sinh(w) \cdot dw$$

Then

$$\int dy = \int \frac{a}{\sqrt{a^2\cosh^2(w) - a^2}} \cdot a \cdot \sinh(w) \cdot dw$$

$$\int dy = \int \frac{1}{\sqrt{\cosh^2(w) - 1}} \cdot a \cdot \sinh(w) \cdot dw$$

From trig identities:

$$\cosh^2 - 1 = \sinh^2$$

$$\int dy = \int a \cdot dw$$

$$y = aw + b$$

Back substituting

$$y = a \cdot \operatorname{arccosh}(\tfrac{z}{a}) + b$$

$$y = a \cdot \operatorname{arccosh}\left(\frac{x+M}{a}\right) + b$$

or

$$\boxed{x = a \cdot \cosh\left(\frac{y-b}{a}\right) - M}$$ Shape of a rope with gravity in the -x direction

$$\boxed{y = a \cdot \cosh\left(\frac{x-b}{a}\right) - M}$$ Shape of a rope with gravity in the -y direction

Plug in constraints:

Left Endpoint: (x0, y0) = (0,1)

$$1 = a\cosh\left(\frac{-b}{a}\right) - M$$

Right Endpoint: (x1, y1) = (1, 1)

$$1 = a\cosh\left(\frac{1-b}{a}\right) - M$$

Length = 2m:

The third equation comes from the total length being 2 meters:

$$\int \sqrt{1 + y'^2} \cdot dx = 2m$$

$$\left(a\sinh\left(\frac{x-b}{a}\right)\right)_0^1 = 2$$

Solving 3 equations for 3 unknowns in MATLAB

```
function J = cost3(z)
 a = z(1);
 b = z(2);
 M = z(3);

 % assume gravity is in the -y direction
 %  y = f(x)

 Length = 2;
 x1 = 0;
 y1 = 1;

 x2 = 1;
 y2 = 1;

 e1 = a*cosh((x1-b)/a) - M - y1;
 e2 = a*cosh((x2-b)/a) - M - y2;
 e3 = a*sinh((x2-b)/a)  - a*sinh((x1-b)/a) + - Length;

 x = [x1:0.001:x2]';
 y = a*cosh( (x-b)/a ) - M;
 plot(x,y);
 pause(0.01);

 J = e1^2 + e2^2 + e3^2;

 end


[A,B] = fminsearch('cost3',10*rand(3,1)-5)

   -0.2296
    0.5000
   -2.0260

  2.1366e-008
```

Result:

$$y(x) = -0.2296\cosh\left(\frac{x-0.5}{-0.2296}\right) - 2.0260$$