Variable Structres Control

NDSU ECE 463/663

Lecture #34

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Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

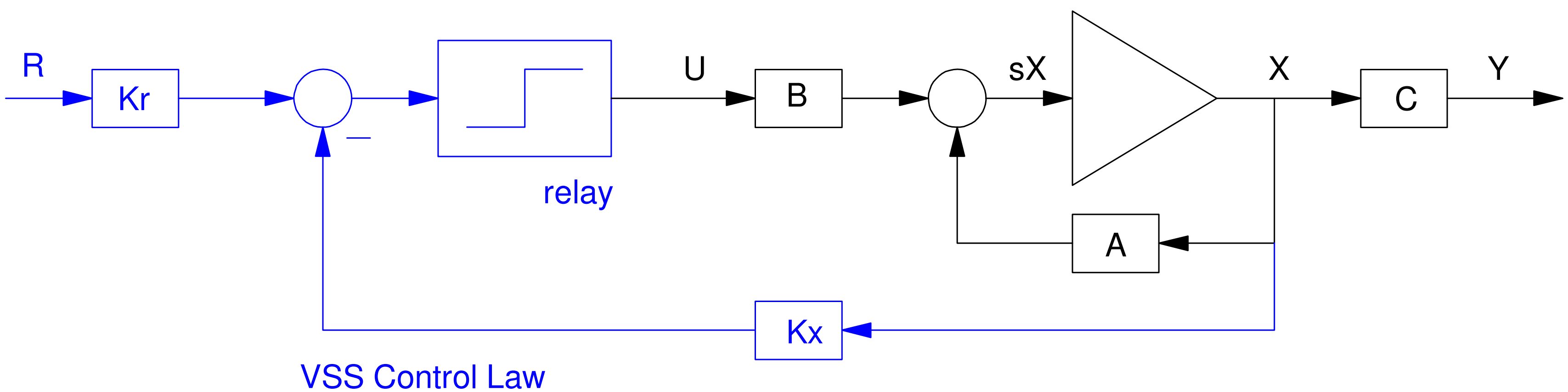
Variable Structures Systems (VSS)

VSS is another control law which uses full-state feedback

$$U = \alpha \cdot sign(K_r R - K_x X)$$

Saturating control is similar:

$$U = \text{limit}(-\alpha, K_r R - K_x X, \alpha)$$



Both of these are nonlinear controllers.

- Eigenvalues don't apply
- Poles and zeros don't apply

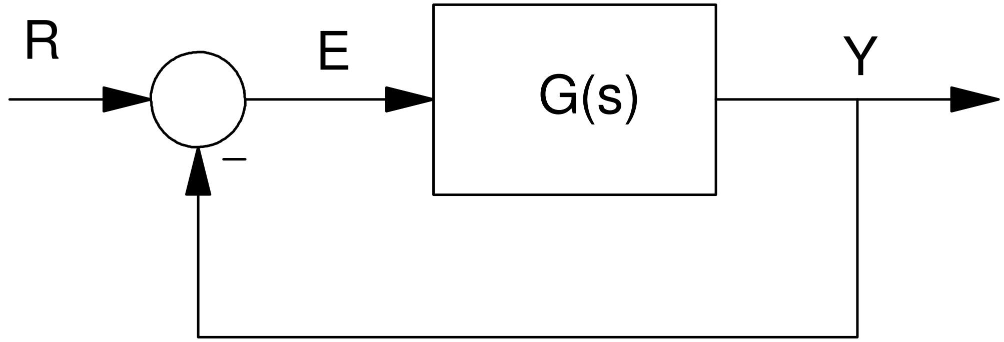
Only three proofs of stability exist for nonlinear systems

- Hyperstability,
- H-infinity, and
- Lyapunov.

H-Infinity:

If the gain is always less than one, the closed-loop system must be stable.

- Useful for analyzing system pertubrations
- As long as the perturbation are small enough, stability won't be affected

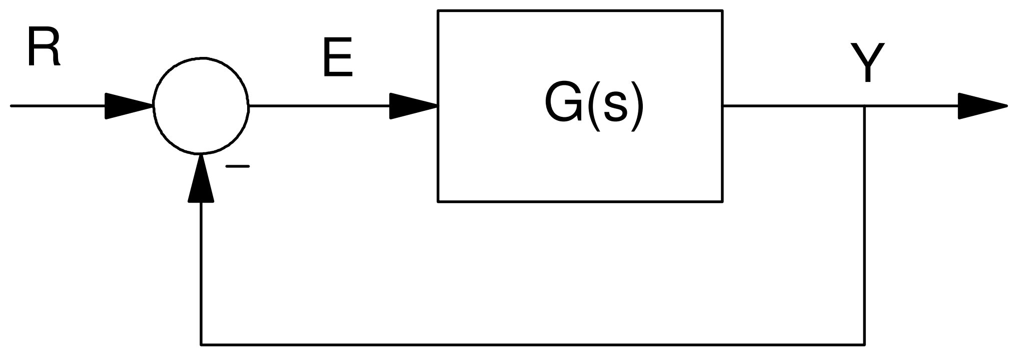


When the phase is 180 degrees, the gain is $a + a^2 + a^3 + a^4 +$

Hyperstability:

If the phase shift of G(s) never reaches 180 degrees, the closed-loop system must be stable.

- You never have positive feedback
- Useful for designing model reference adaptive controllers



Lyapunov Stability:

- Define an energy function which is positive definite.
- If you can show the change in energy is always negative definite, the system must be stable.

Example 1: Use Lyaponov methods to prove the following system is stable:

$$\dot{x} = -3x$$

Step 1: Define a positive definite energy function:

$$V = \tfrac{1}{2}x^2$$

Step 2: Check that the change in energy is negative definite:

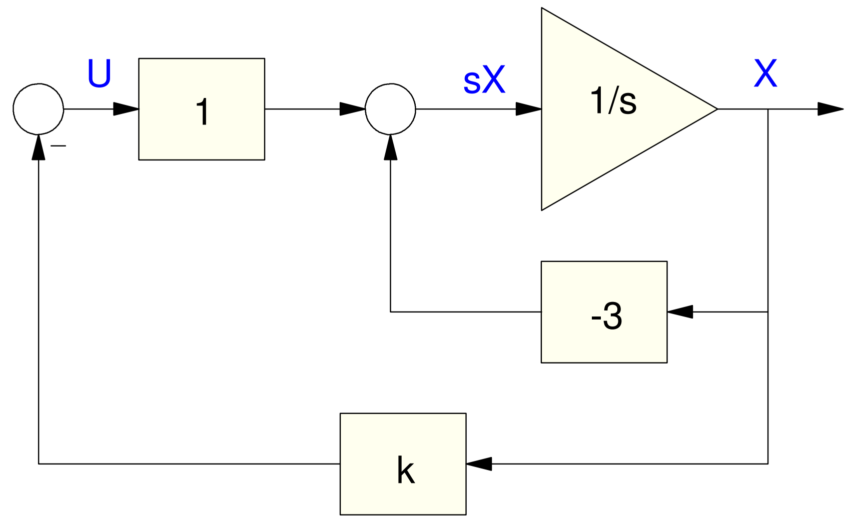
$$\dot{V} = x\dot{x} = x(-3x) = -3x^2 < 0$$

This system is stable.

Example 2: Find the range of k which results in a stable system:

$$\dot{x} = -3x + u$$

$$u = -kx$$



Step 1: Define an energy function:

$$V = \tfrac{1}{2}x^2$$

Step 2: Check that the change in energy is negative definite:

$$\dot{V} = x\dot{x}$$

$$\dot{V} = x(-3x - kx)$$

$$\dot{V} = -(3 + k)x^2$$

To be stable

$$3 + k > 0$$

$$k > -3$$

Example 3:

$$\dot{X} = AX + BU$$

Define a sliding surface

$$\sigma = CX$$

Define an energy function

$$V = \tfrac{1}{2}\sigma^T\sigma > 0$$

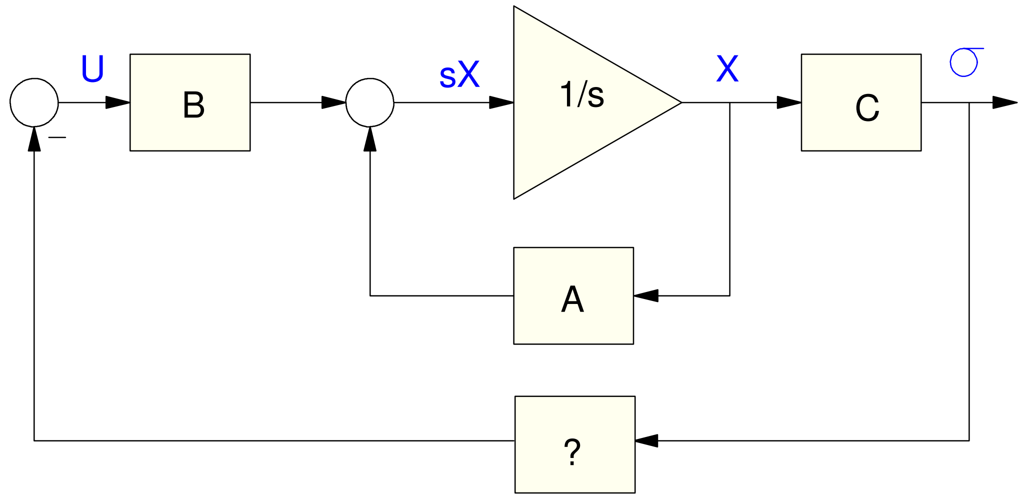
Pick U so that $\dot{v}$ is negative definite:

$$\dot{V} = \sigma^T\dot{\sigma} < 0$$

Substituting:

$$(CX)^T\left(C\dot{X}\right) < 0$$

$$X^TC^T(CAX + CBU) < 0$$



If

$$|CBU| > |CAX|$$

$$CB > 0$$

then

$$X^T C^T (CBU) < 0$$

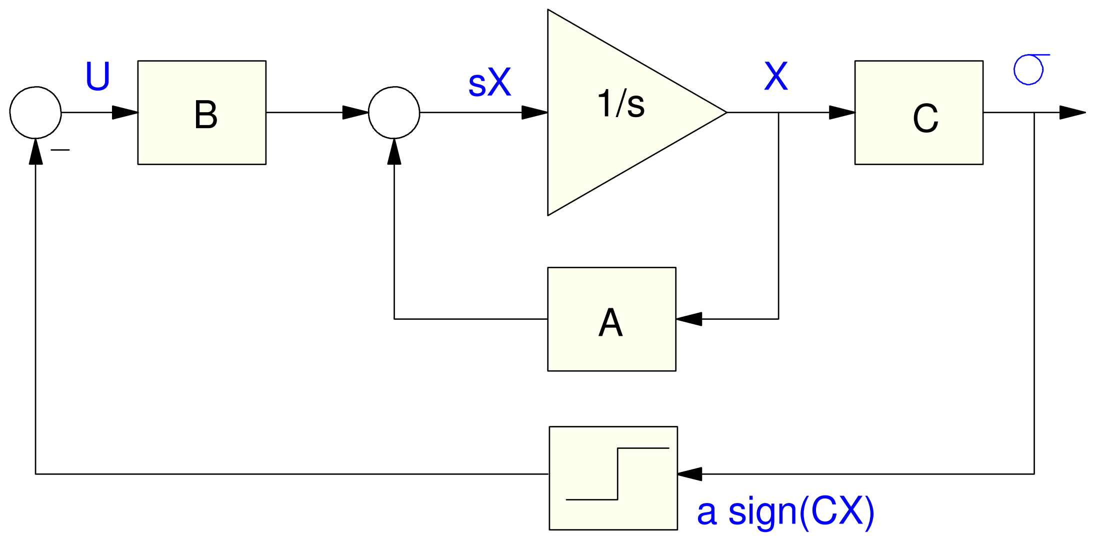
$$X^T C^T U < 0$$

Let

$$U = -\alpha \cdot sign(CX)$$

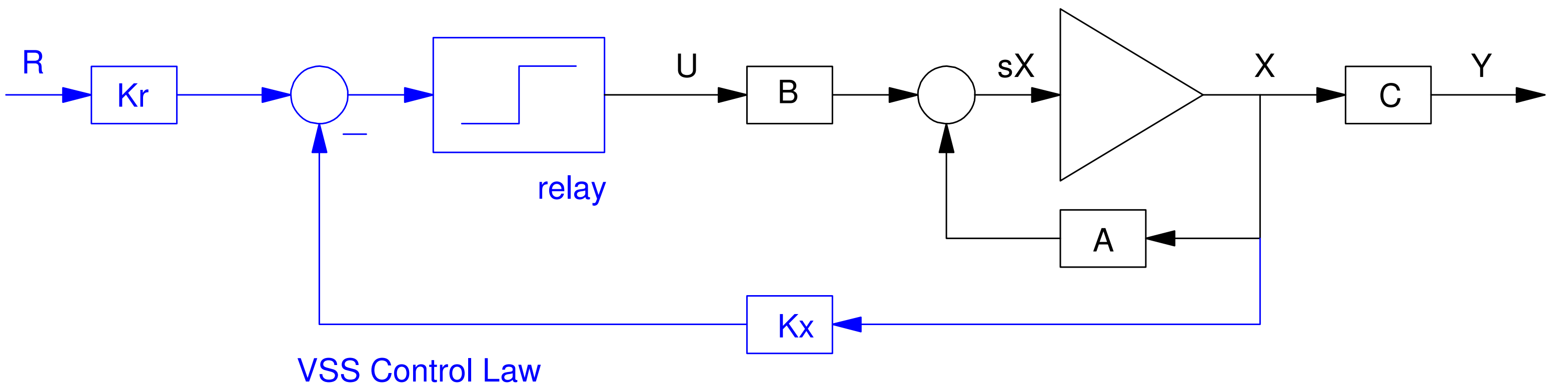
where

$$|CB\alpha| > |CAX|$$



If you add in a set point (R), you get

$$U = \alpha \cdot sign(K_r R - K_x X)$$



Example: Double Integrator:

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

Define the sliding surface to be

$$\sigma = \begin{bmatrix} 1 & 1 \end{bmatrix} X$$

Assume X is bounded by 10

$$|CBU| > |CAX|$$

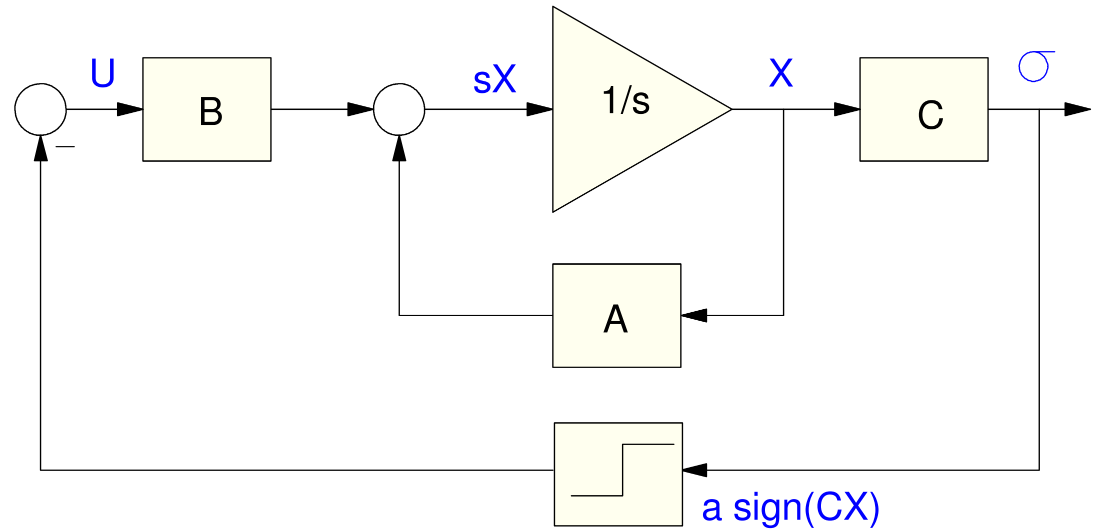
$$|\alpha| > 10$$

Then

$$U = -10 \cdot sign(CX)$$

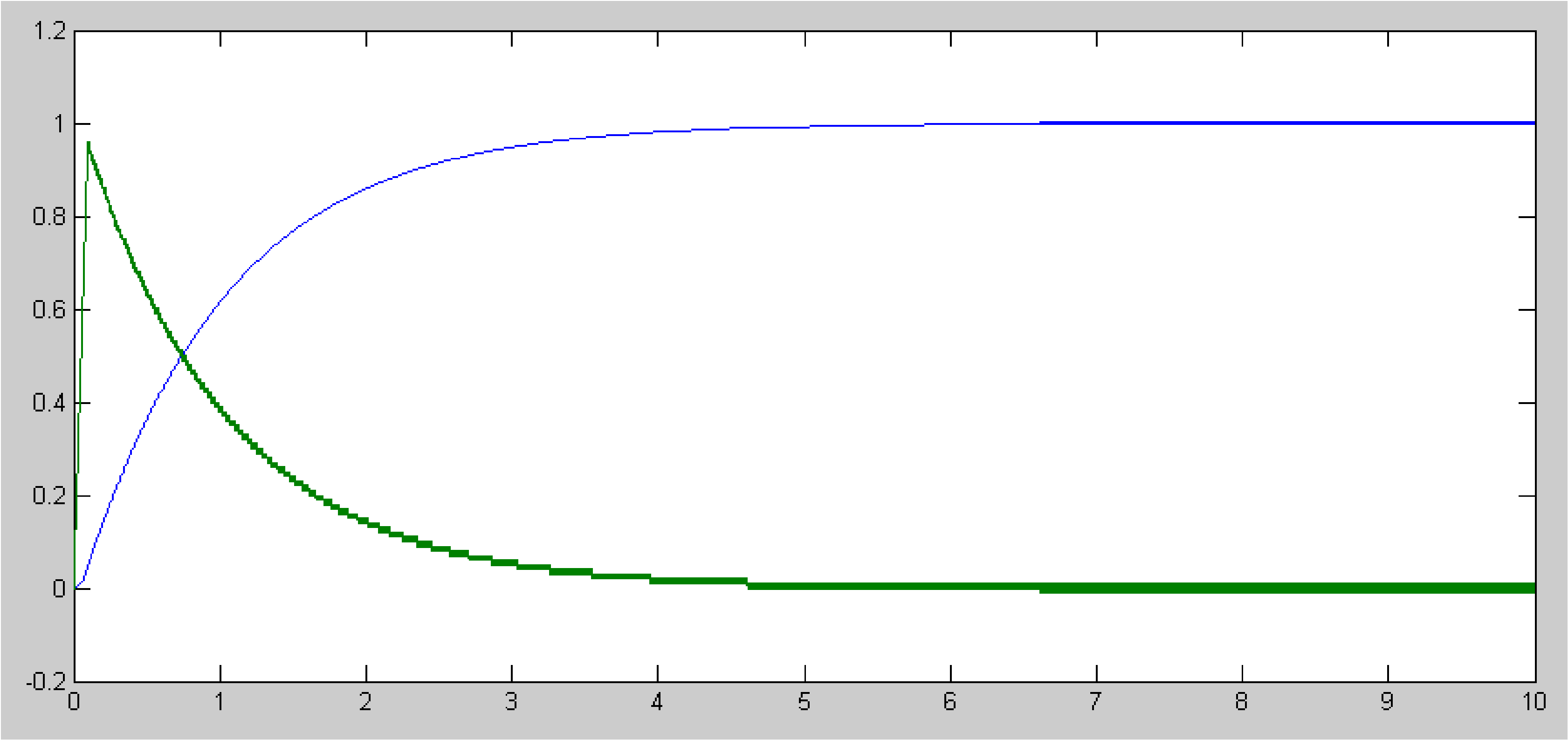
Adding in a reference

$$U = -10 \cdot sign((x - R) + (\dot{x}))$$



System Response:

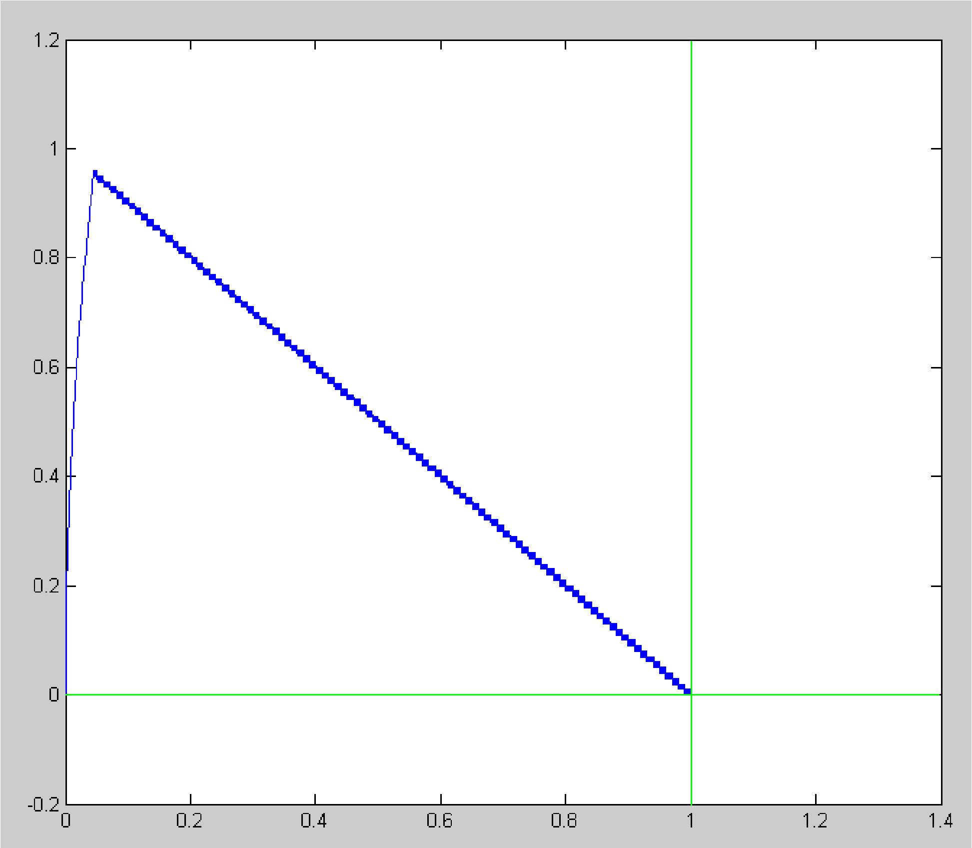
- Note that the system behaves like a system with a pole at -1
- (the zero in the transfer function from R to σ)



Step Response for a VSS controller with σ = (s+1)X

Phase Plane:

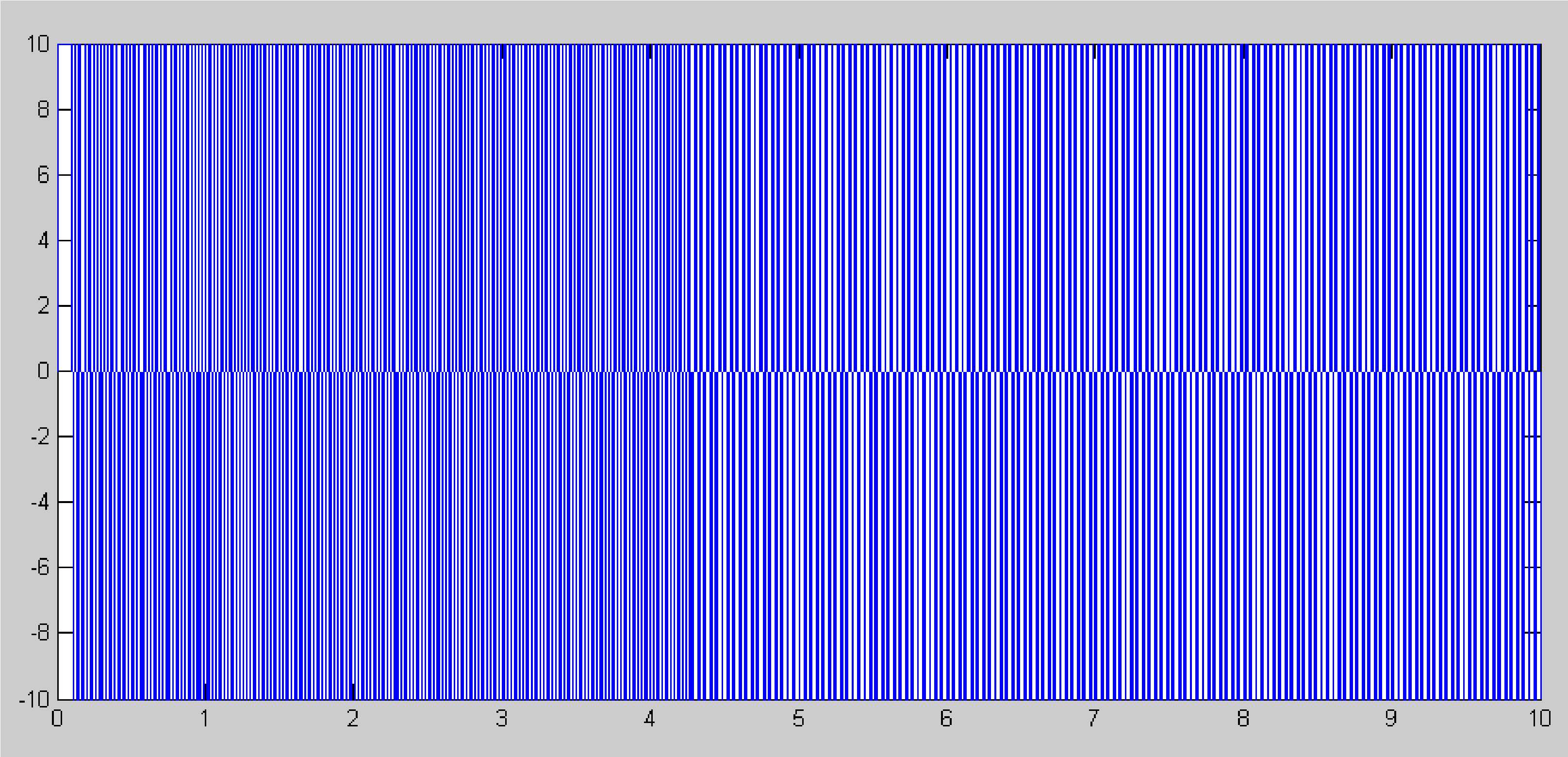
- Plot $\boldsymbol{x}$ vs. $\dot{\boldsymbol{x}}$
- Shows eigenvector for s = -1



Phase Plane for y = (s + 1) x along with its sliding surface

Problem: The input chatters

- a.k.a. Bang Bang Control



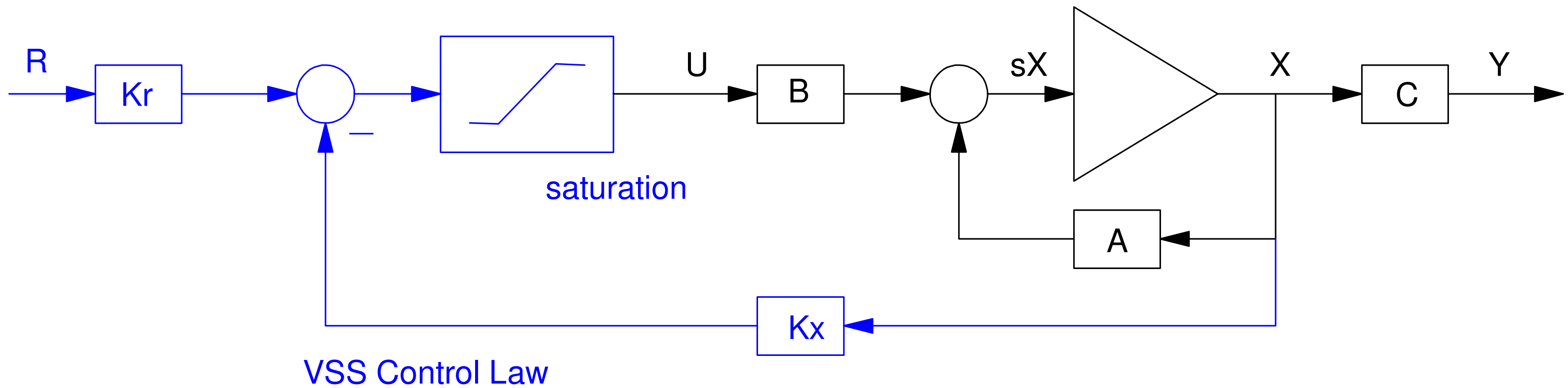
Input u(t). Note that it chatters from -10 to +10 while you're on the sliding surface.

Saturating Control

Rather than using a relay function, a saturating function with a large gain results in

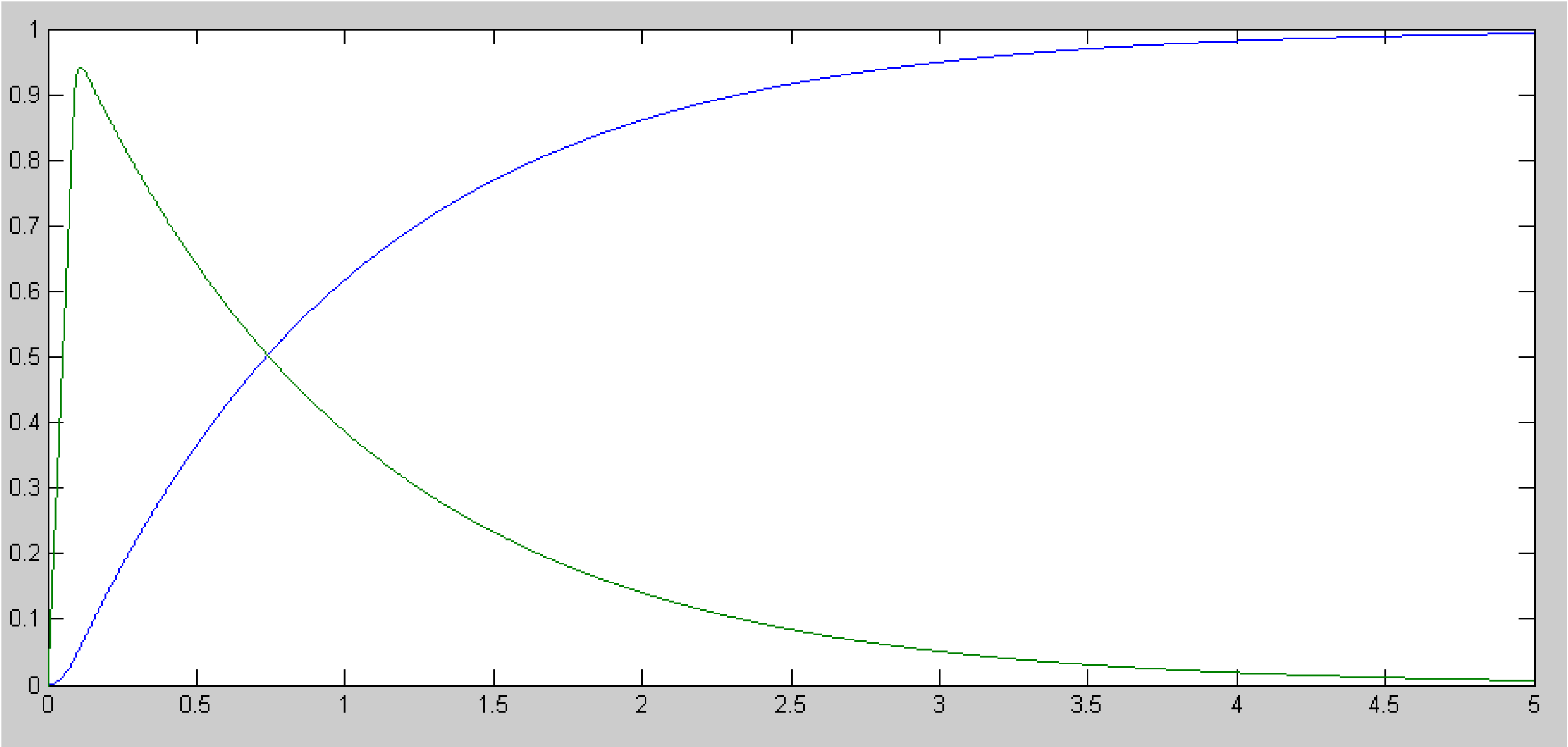
- Almost the same result (same sliding surface, same closed-loop response), but
- The input no longer chatters

$$U = \mathrm{limit}(-10, k(K_r R - K_x R), 10)$$



Result:

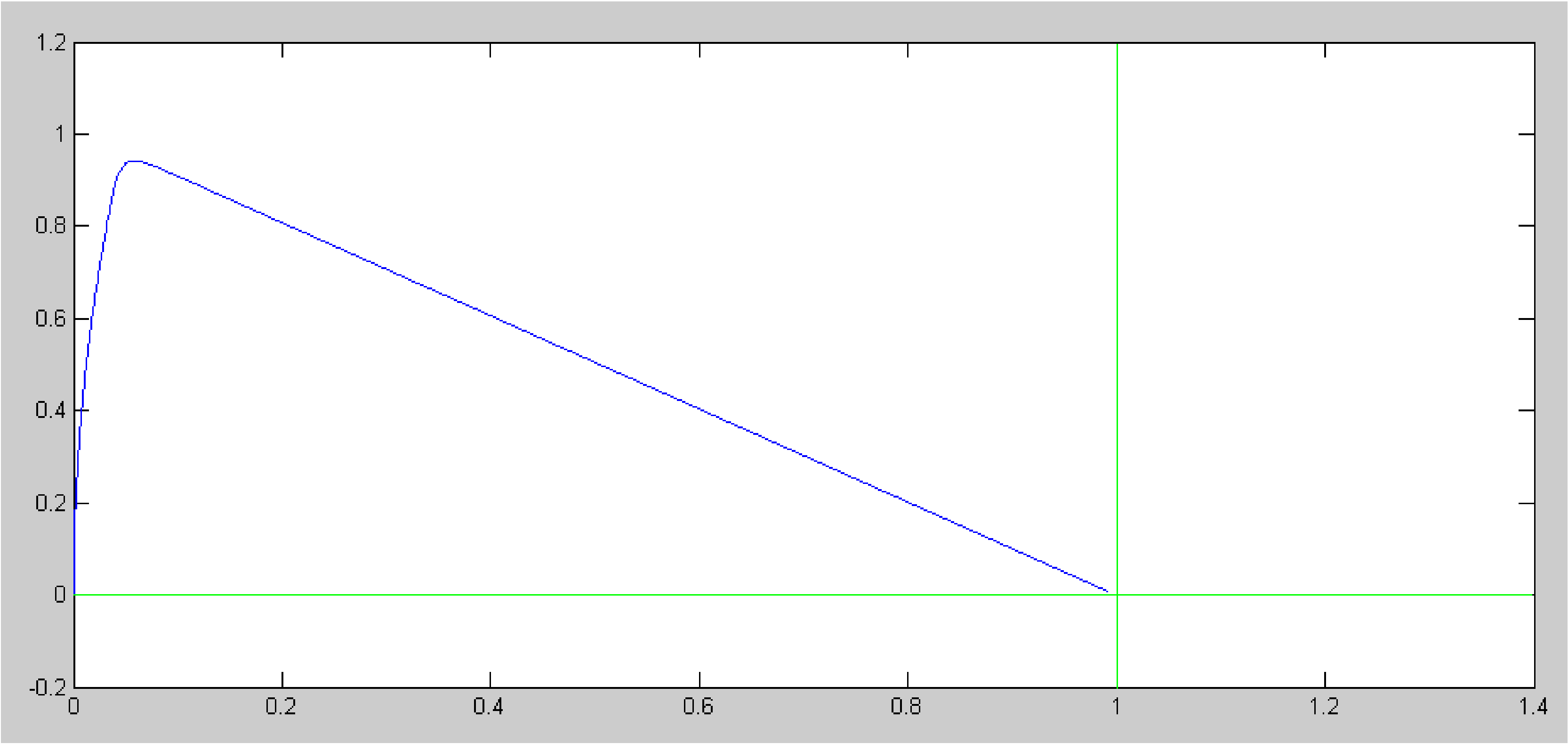
- Almost the same
- Approach the set point as s = -1



x (blue) and dx (green).
Saturating Control with sigma = (s+1)x

Phase Plane

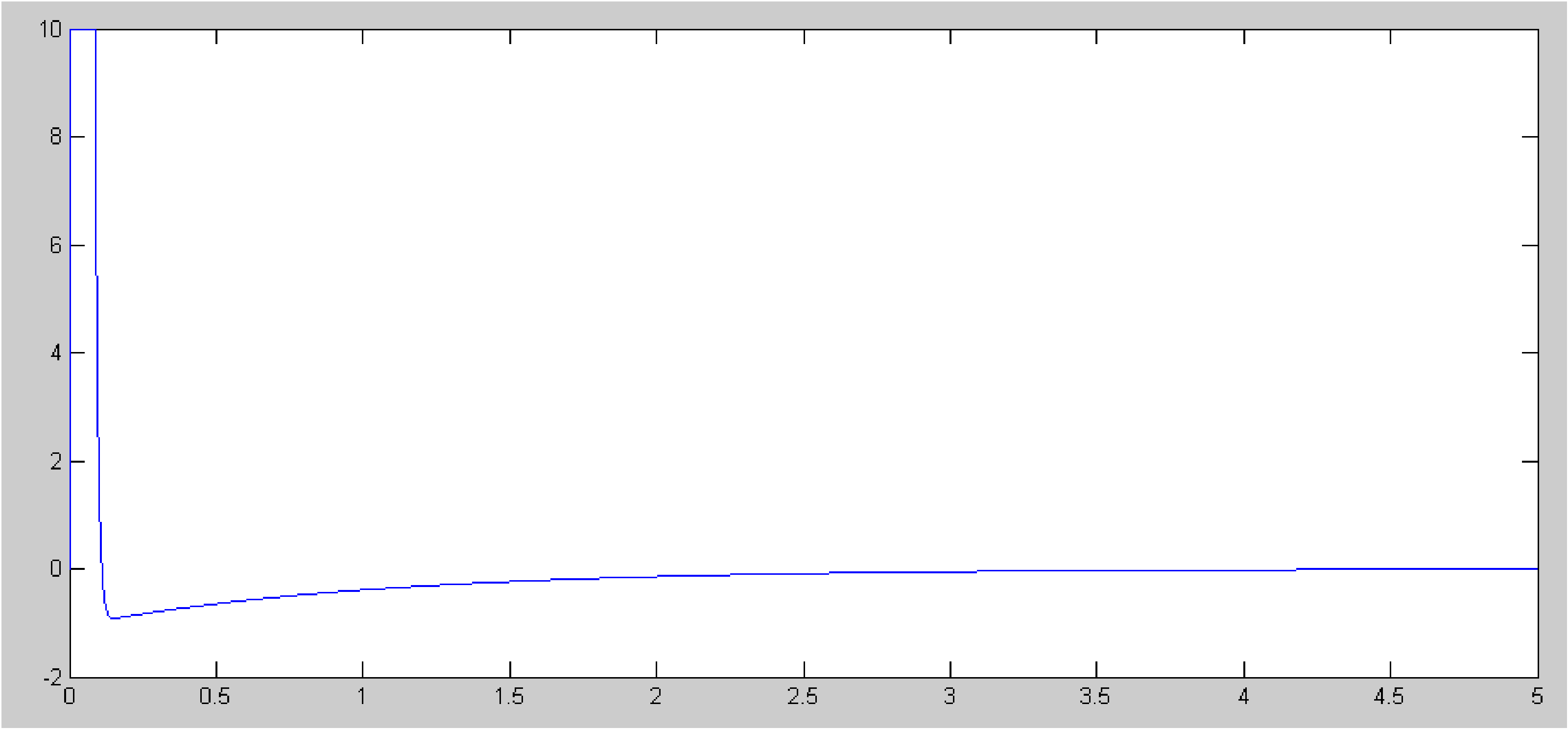
- Approach the eigenvector (sliding surface) at s = -1



Phase Plane: Saturating Control with sigma = (s+1)x

But, the input no longer chatters.

- Note: the zeros determine the sliding surface and the closed-loop poles



Input (U) for saturating control with sigma = (s+1)x

VSS Control for an RC Filter (real zeros)

Assume 4-stage RC filter:

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]
B = [1;0;0;0]
```

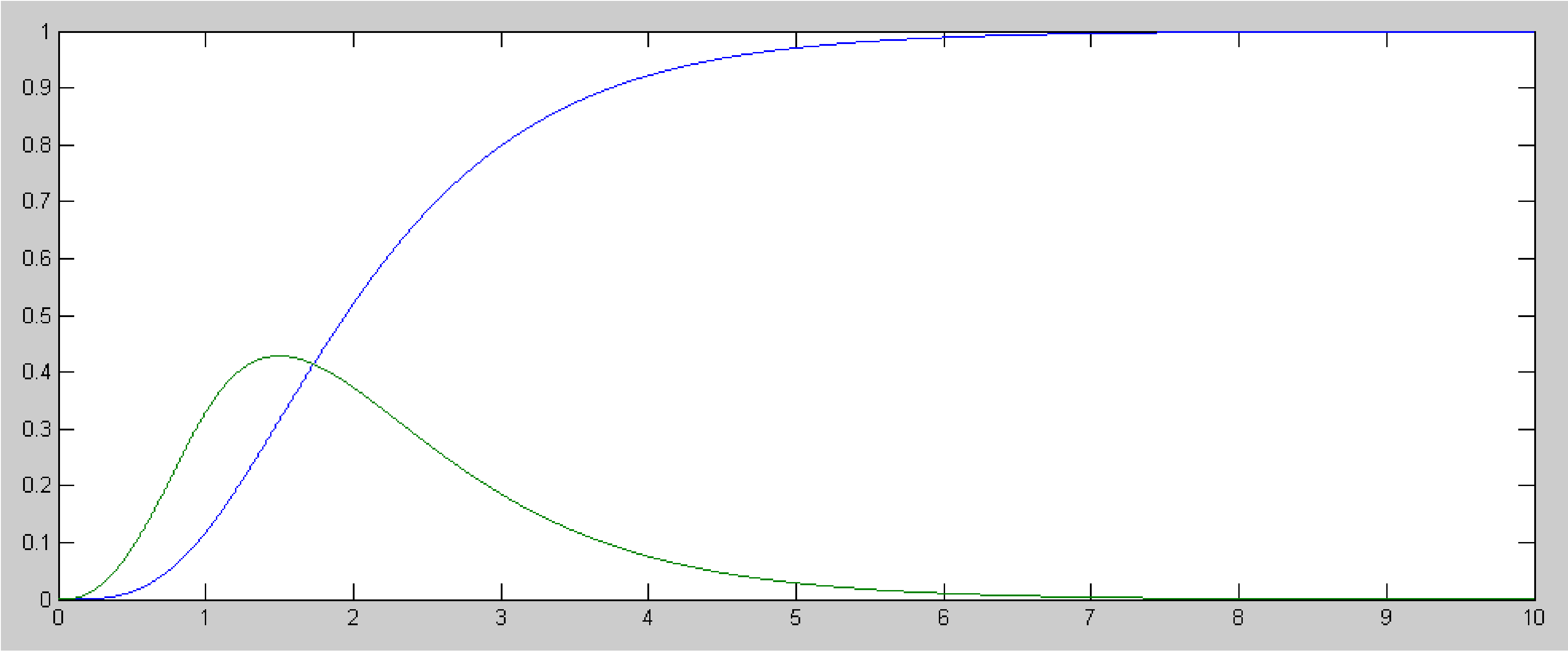
To place the zero, convert to controller canonical form using Bass Gura:

```
Cz =     6     11      6      1

Kx = Cz*inv(T)

Kx =    1.0000    1.0000    2.0000    2.0000
```

Check that the zeros are at {-1, -2, -3}

```
eig(A-100*B*Kx)

 -101.0203
   -2.9899
   -1.9898
   -1.0000
```

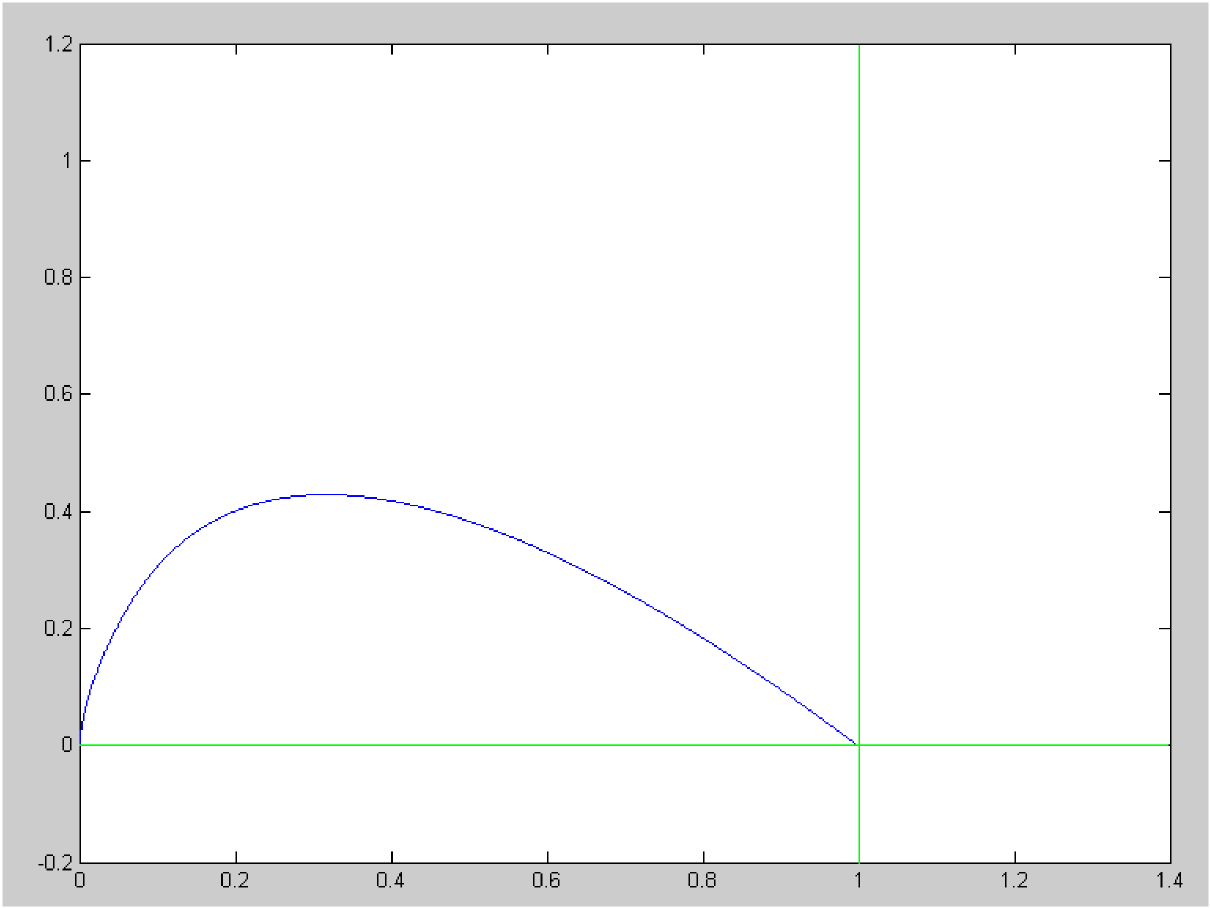
Step Response:



Step Response with VSS Control: sigma = (s+1)(s+2)(s+3)x
Position (blue) and velocity (green)

Phase Plane:

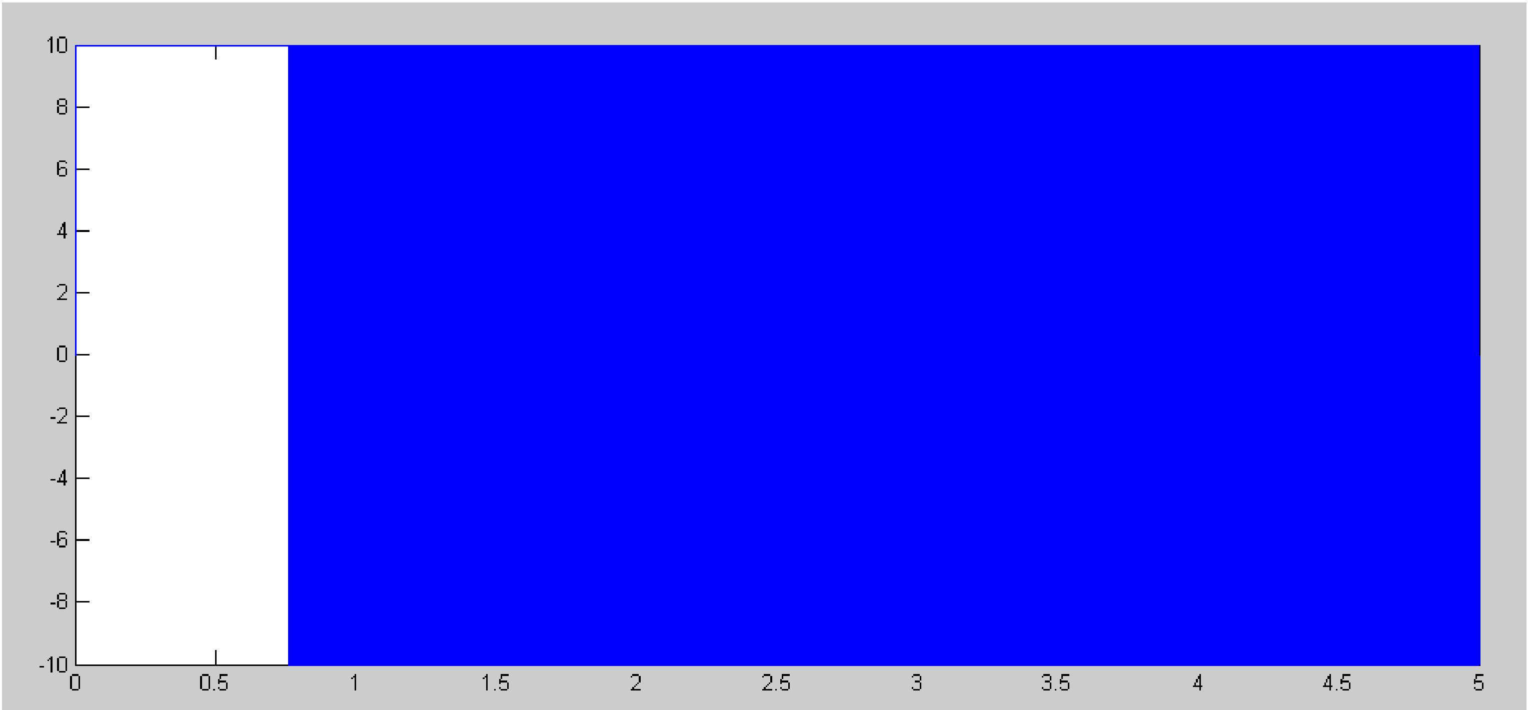
- Shows the dominant pole at s = -1:



Phase Plane: position vs. velocity

The input chatters

- bang-bang control



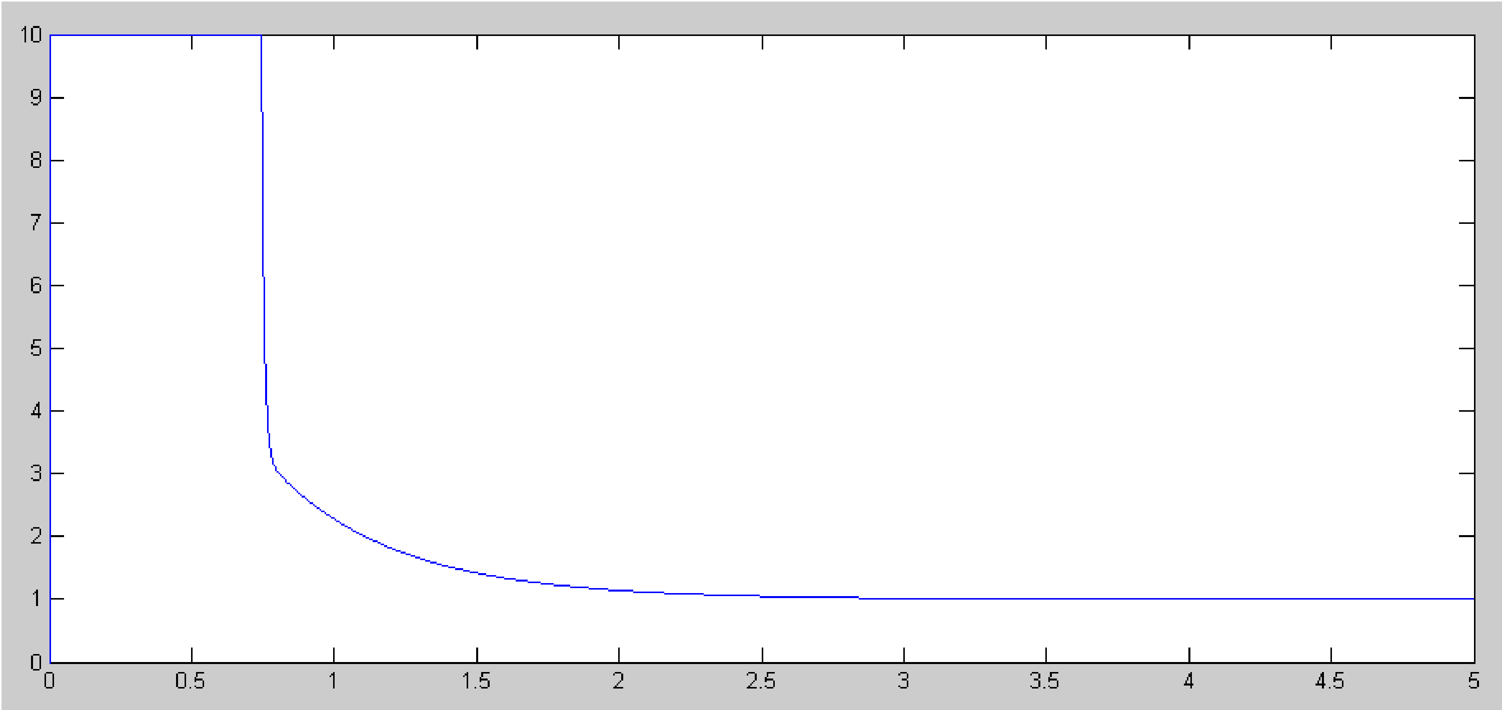
Input u(t) with VSS Control. Once you hit the sliding surface, it chatters between -10 and +10.

Case 2: Saturating Control.

$$U = 10 \cdot sign(K_r R - K_x X)$$

becomes

$$U = \text{limit}(-10, 100(K_r R - K_x X), 10)$$

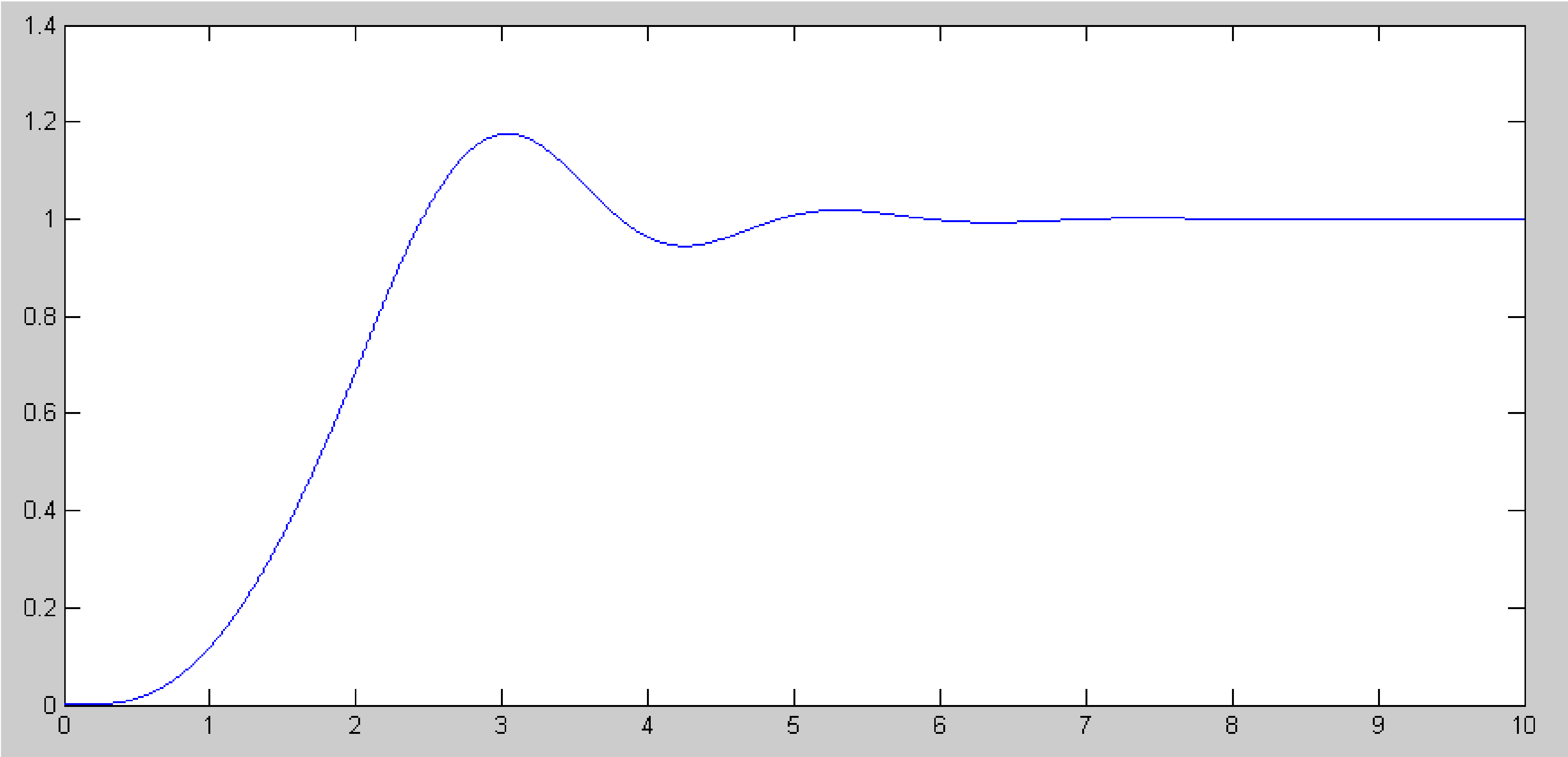


Input u(t0 with Saturating Control. Once you approach the sliding surface, u(t) stops clipping.

VSS with complex zeros

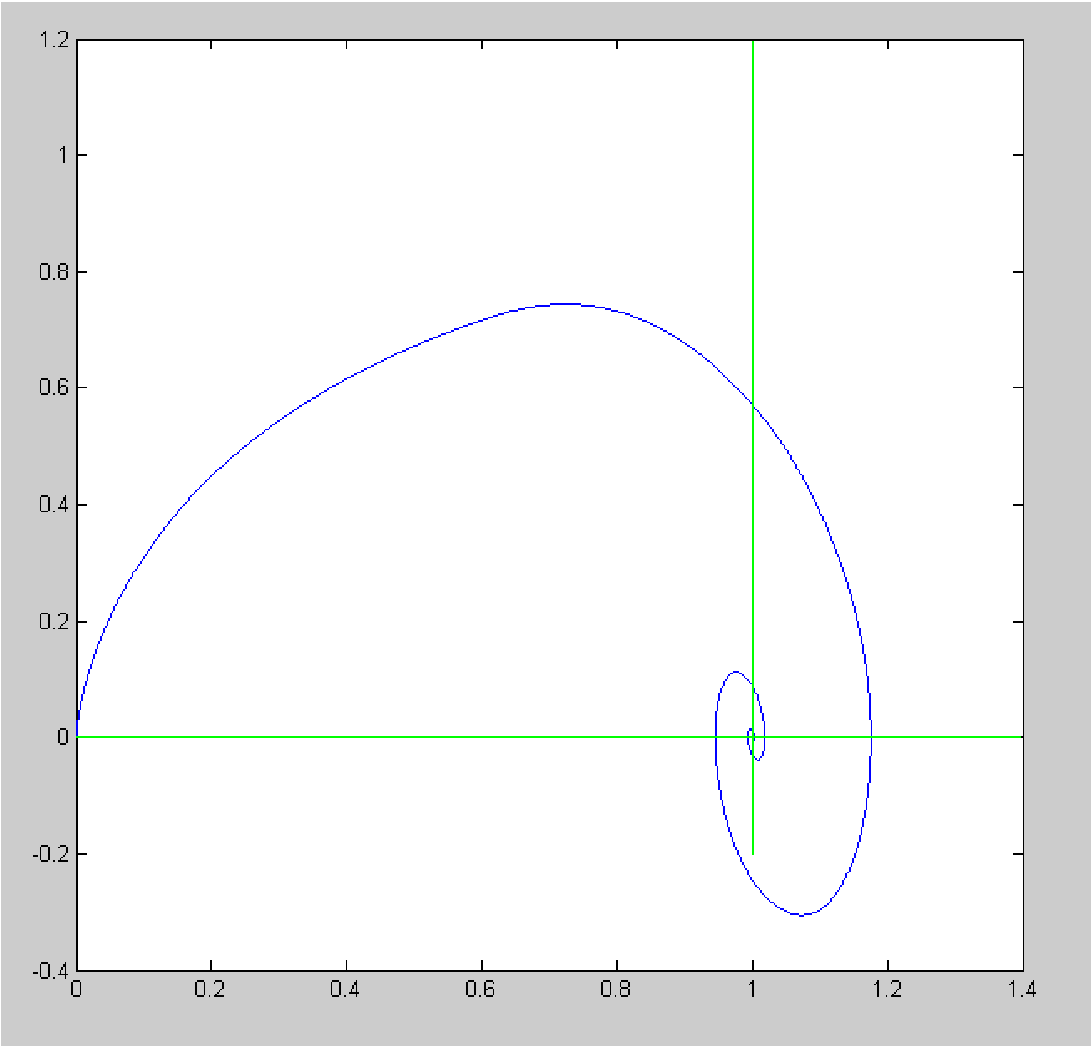
- Pick Cz to place the zeros
- Can be real or complex

```
poly([-1+j*3,-1-j*3,-3])

     1     5     16     30

Kx = [30, 16, 5, 1]*inv(T)

    1.0000    0.0000   10.0000   19.0000
```



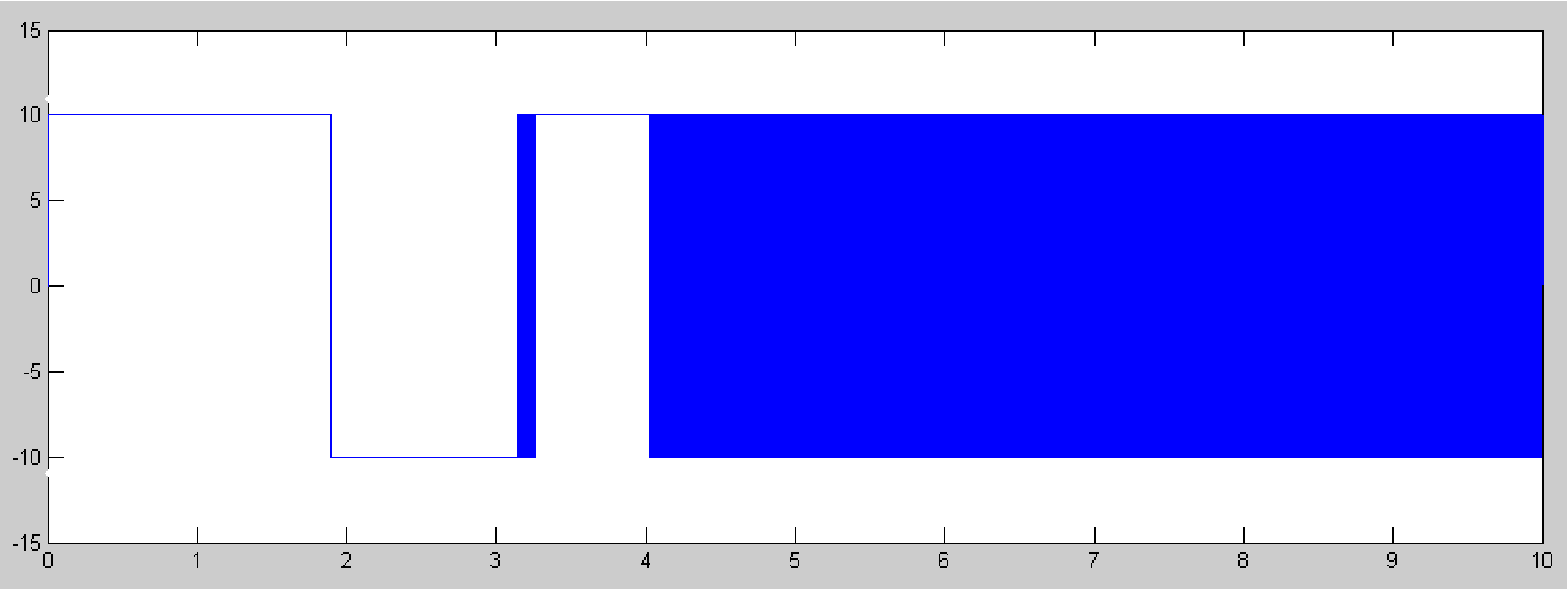
VSS Control: Step Response with zeros at {-1 + j3, -1 - j3, -3}

The phase plane is from



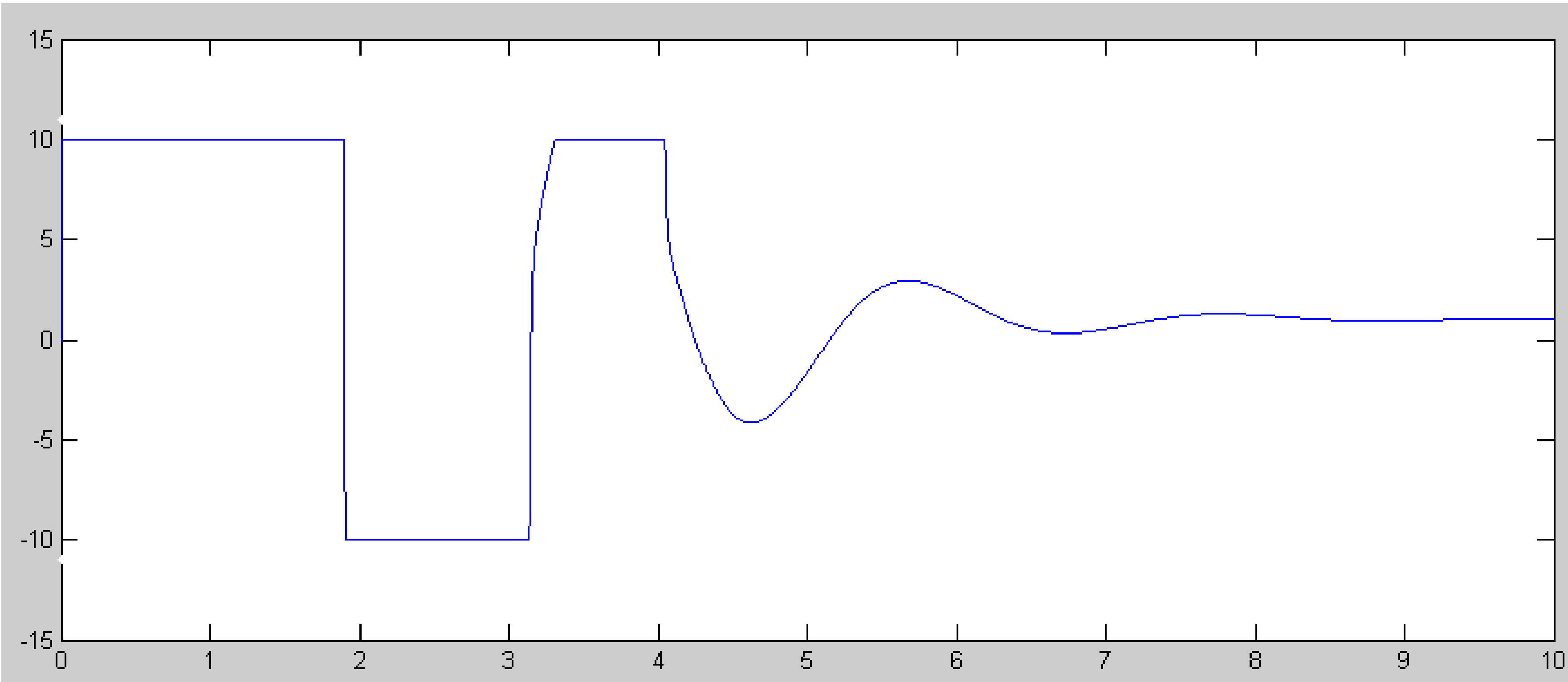
Phase Plane: The log spirals correspond to the complex zeros at {-1 + j3, -1 - j3}

Input chatters when on the sliding surface



Input, u(t), for VSS control with zeros at {-1 + j3, -1 - j3, -3}. Once you hit the sliding surface, the input chatters between -10 and +10.

If you change to a saturating controller, the response is almost the same except that the input no longer chatters:



Input, u(t), for a Saturating Control with zeros at {-1 + j3, -1 - j3, -3}. Once you hit the sliding surface, the input drops between -10 and +10.

Summary

VSS & Saturating Control are a form of full-state feedback

- $U = -K_x X$ becomes $\sigma = CX$ and $U = -f(\sigma)$
- Zeros of transfer function determine the closed-loop poles

These controllers have some nice properties

- Changes in the system dynamics don't affect the closed-loop response
  - *assuming the zeros don't change*
- The closed-loop system behaves as a lower-order system
  - *System order reduced by one*
- The input is easy to implement
  - *Slam to* $\pm\alpha$ *(VSS)*

A saturating controller is just a VSS controller with a saturating funciton

- $U = -\alpha \cdot sign(\sigma)$ VSS
- $U = -\alpha \cdot \mathrm{limit}(-1, -k\sigma, +1)$ Saturating