

## CHAPTER 8 ARTERIAL MODELS

The purpose of this chapter is to expose the cardiovascular engineer to lumped parameter models of the arterial system. The rationale is that just as we built cardiac models to better help us understand function and obtain quantitative metrics concerning cardiac function, so too will we build arterial models to improve our understanding and obtain quantitative metrics about the performance of the arterial system. Once we have adequate models of the heart and arterial system, we can begin to understand their complex interaction

### Chapter 8 learning objectives:

1. Basic hemodynamic Properties (resistance, capacitance and inductance)
2. Two and n-element Windkessels
3. Fourier Series
4. Input Impedance
5. Characteristic Impedance
6. Methods of parameter estimation
7. Future work

### 8.1 Basic Hemodynamic properties

Ultimately the problem of developing a model of the arterial system reduces to figuring out what's inside of "the black box" while only knowing the inputs to the black box (Figure 1). There are many approaches to solving the mystery of what's inside the box and we'll explore a few of these in this chapter.

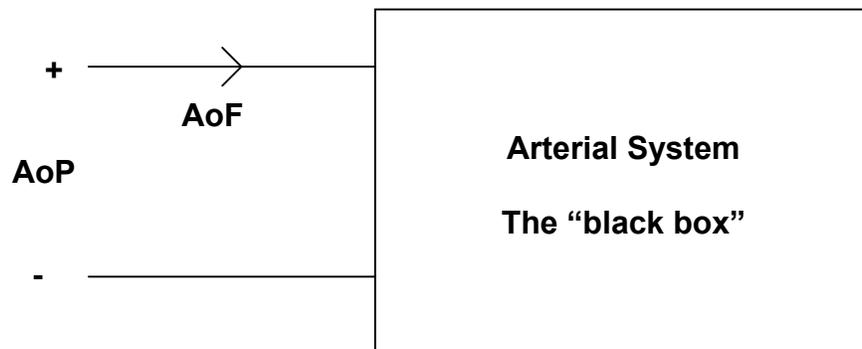


Figure 8-1 Arterial System

Perhaps one of the simplest ways to view the arterial system is that used most by clinicians. If we measure the mean arterial pressure and the cardiac output, we can calculate a relationship between mean arterial pressure and flow. This ration

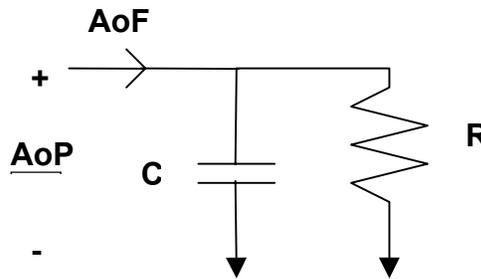
## 8-2

is termed total peripheral resistance (TPR) or systemic vascular resistance (SVR).

$$SVR = TPR = \frac{\text{Average Arterial Pressure}}{\text{Cardiac Output}} \approx \frac{\overline{AoP}}{CO}.$$

In this model the black box is modeled as a resistor and we can calculate the resistance. This information can be used to deduce whether the arterial resistance is elevated.

The validity of this model can be checked by injecting the time domain flow into the model and calculating the resulting pressure and comparing it to the measured aortic pressure. One can easily understand that the aortic flow (AoF) and the aortic pressure (AoP) are not linearly related and that another model element must be present. We are well aware that the arterial system exhibits elasticity and thus another element to include in the model is compliance. This approach results in the well-known 2-element windkessel model, Figure 8-2



**Figure 8-2 Windkessel Model of Arterial System**

The way a windkessel works is that as flow is ejected from the heart, it can either flow into the capacitor or down the resistance. When the heart stops ejecting, the pressure in the arterial system is maintained because there is a volume in the capacitor. As some of the volume drains from the capacitor through the resistance, the pressure drops in the arterial system until another volume is ejected from the heart.

Many ways have been devised to estimate the arterial parameters from this model. We'll investigate one of these methods.

We can write a current, or conservation of mass, equation:

$$AoF = C \frac{dAoP}{dt} + \frac{1}{R} AoP$$

This equation indicates that the flow that the heart ejects will either flow into the capacitor or through the resistance. At this point we have one equation, but two

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unknowns. Many methods have been devised to obtain the second equation, but we'll focus on but one of these.

To obtain the second equation we can integrate the flow equation over two intervals.

$$\int_{t_0}^{t_1} AoF dt = C \int_{t_0}^{t_1} \frac{dAoP}{dt} dt + \frac{1}{R} \int_{t_0}^{t_1} AoP dt$$

$$\int_{t_1}^{t_2} AoF dt = C \int_{t_1}^{t_2} \frac{dAoP}{dt} dt + \frac{1}{R} \int_{t_1}^{t_2} AoP dt$$

Now we have two equations and two unknowns and can solve for the two arterial parameters, R and C. Let's investigate these equations further. If we integrate over just the ejection period we obtain:

$$SV = C[AoP(t_2) - AoP(t_1)] + \frac{1}{R} \int_{t_1}^{t_2} AoP dt$$

Which can be thought of in terms of a volume balance.

$$SV = \Delta V_{capacitor} + \Delta V_{resistor}$$

Thus, the stroke volume either fills the capacitor or runs through the resistance. Interestingly, if we have a steady beat (i.e.  $[AoP(t_2) = AoP(t_1)]$ ) then the two-element equation reduces to the clinical definition of TPR with just a minor manipulation.

$$\frac{SV}{T} = \frac{1}{T} \frac{1}{R} \int_{t_1}^{t_2} AoP dt$$

$$\overline{AoF} = \frac{\overline{AoP}}{R},$$

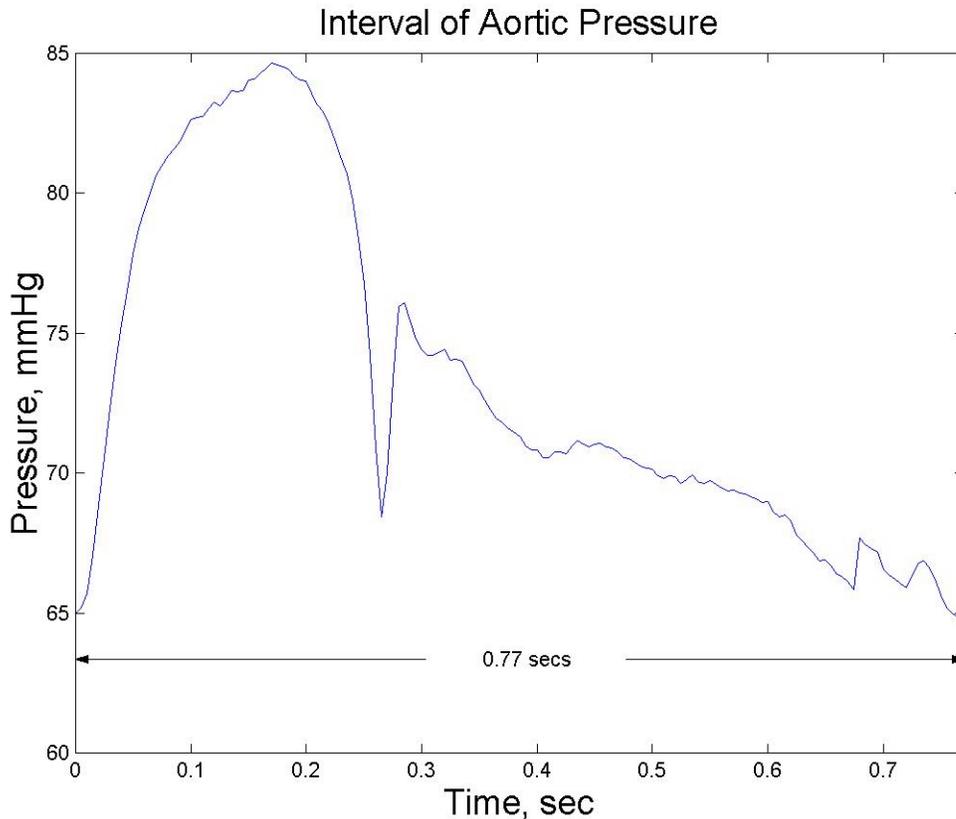
where T is the time period where  $[AoP(t_2) = AoP(t_1)]$ .

Once again we can check the validity of this model by injecting flow into the circuit and calculating the resulting pressure and comparing it to measured AoP. If we do this, we see that it is much better than a pure resistance, but it still could use some improvement. This begins the "great element chase". As we add more elements we get better agreement between calculated and measured AoP, but we also need more equations and measurements to estimate the parameters of these higher order models. In literature, one can find three element models, four element models and so on. As we increase the number of model elements we begin to move away from "lumped" parameter models into distributed models. The performance of the distributed models is better but at the cost of many equations, many measurements and less easily understood physiological meaning. Is there another way?

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Let's take a different approach. Going back to Figure 8.1, let's assume we just have aortic pressure and aortic flow. Further, let's assume we cannot inject any other signals into the black box. Normally, to determine what's in a black box, we might inject a step input and measure the response (stimulus/response). Or we might inject a swept frequency sinusoid and measure the frequency response. However, we've assumed that we cannot inject any new signals into the black box. How do we figure out what's in the box for any arbitrary waveform? Many methods exist to do this, but we'll concentrate on one.

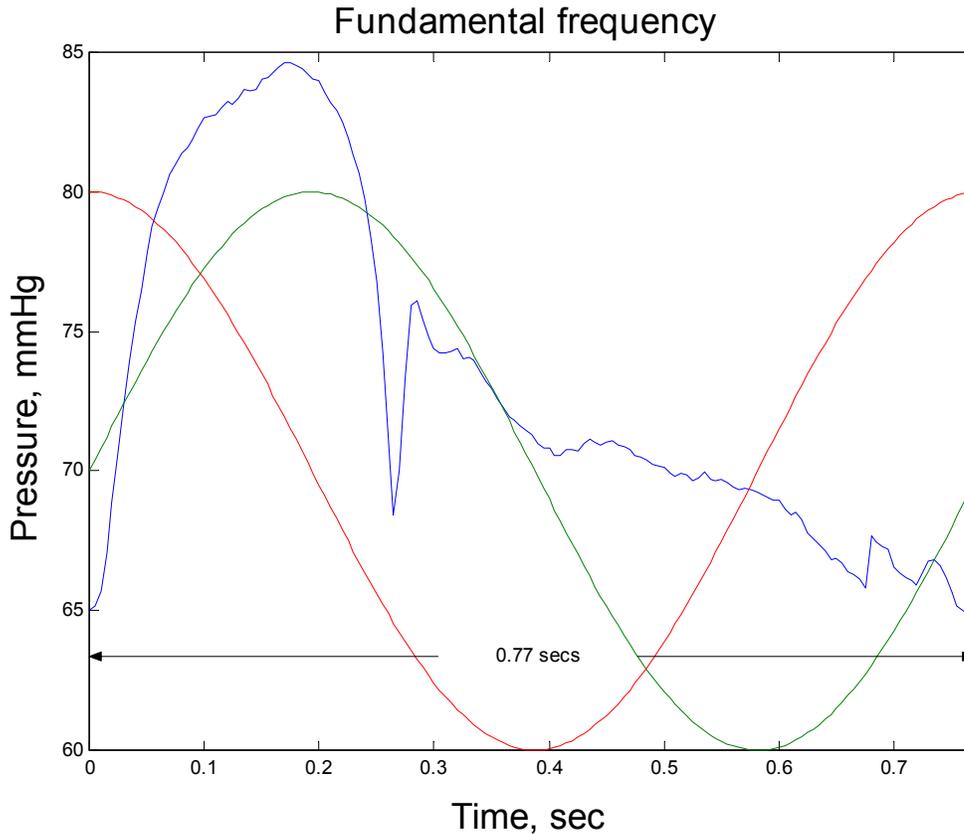
We can take any arbitrary waveform and make it into a series of cosines and sine wave. This is called a Fourier series. If we have enough terms in the Fourier series we can exactly duplicate any arbitrary waveform. Any arbitrary harmonic waveform can be comprised of an average and a series of sine and cosine waves of varying amplitudes. First, let's look at some examples, Figure 8-3.



**Figure 8-3 Selected Interval of Aortic Pressure**

We can find the frequencies of interest for any waveform by using a very simple method. First, we take an interval of data. For most hemodynamic analysis this interval will be a beat. Here the interval is 0.77 seconds. Next, what is the smallest frequency sine wave that we can fit into that interval? It is a sine wave that completes its circumnavigation of the unit circle in 0.77 secs, any slower than this and we wouldn't have a complete sine wave. Thus the frequency,  $f$ , of

the slowest sine wave that fits into our selected interval is  $1/0.77$  secs, or about 1.3 Hz (Figure 8-4). Armed with this information we can construct a table (Table



**Figure 8-4 Lowest Frequency Sine and Cosine Wave to Fit Into Interval**

8-1). In the first column, time is listed at each sample time. The corresponding pressure measurement is also listed in the next column. In the third column we multiply the pressure times a cosine wave of the smallest frequency. In the fourth column we multiply the pressure times a sine wave of the smallest frequency for our window. In subsequent columns we multiply pressure and cosine and sine waves of twice the smallest frequency (a harmonic), then three times the smallest frequency and so on.

Next to determine the average of the waveform we simply sum the pressure column and divide by the number of samples. To determine the coefficients of the sine and cosine waves for each of the frequency terms we simply sum that particular column and divide by the one-half the number of samples. Repeating this procedure for a number of frequencies results in the equation that will allow us to represent the waveform by an average and a sum of cosines and sines as shown below Table 8-1. We determine the A's and the B's by summing the appropriate column and dividing by the one-half number of samples. What this means is that any waveform can be thought of as a signal generator which is made from a battery (average term) and many sinusoids of many frequencies in series with the battery.

We can simplify the procedure somewhat by combining terms of the same frequency.

$$M_i = \sqrt{A_i^2 + B_i^2}$$

$$\phi_i = \tan^{-1}\left(\frac{B_i}{A_i}\right)$$

Thus we can replace the cosine and sine term with a single frequency source with magnitude  $M$  and phase,  $\phi$ . This leads us to the concept of a phasor. A phasor has a magnitude and an angle. We can depict this on a unit circle as shown below in Figure 8-5.

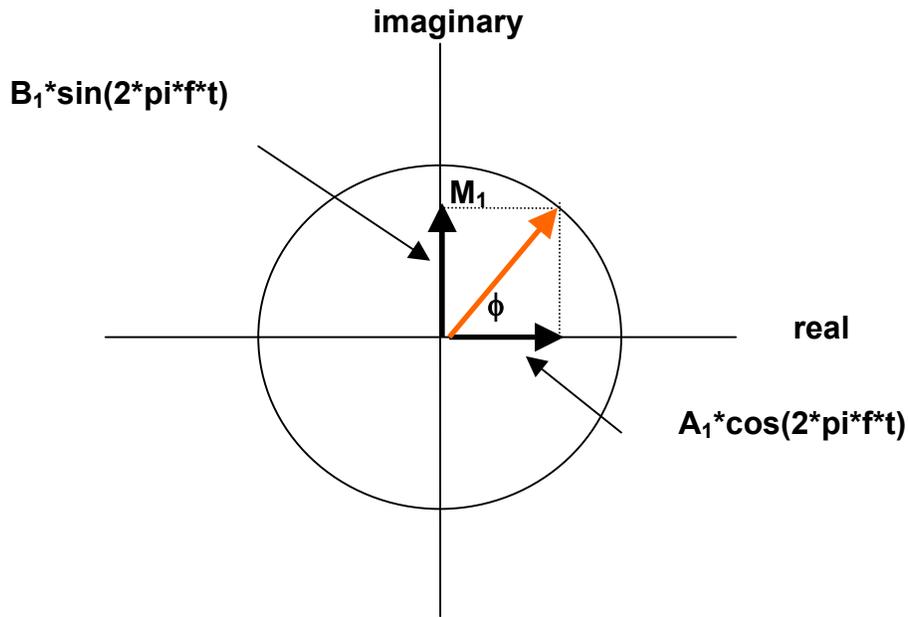


Figure 8-5 Phasor Representation

Thus, ultimately we can think of most any hemodynamic signal as a sequence of other signals. For short hand, we can write an expression for the phasor in Figure 8-6.

$$Y_1 = M_1 \cos(2 * \pi * f * t + \phi_1)$$

$$\vec{Y}_1 = M_1 \angle \phi_1$$

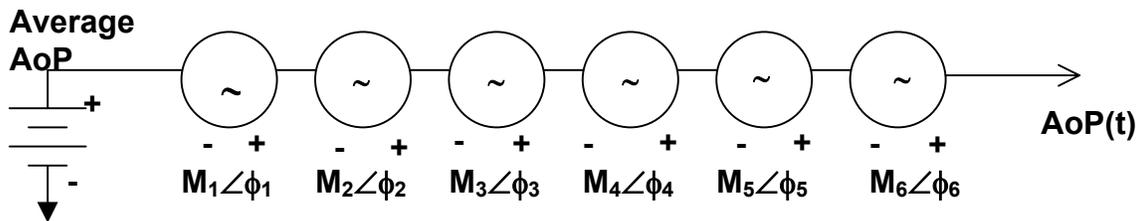


Figure 8-6 Series Addition of Signals

TABLE 8-1

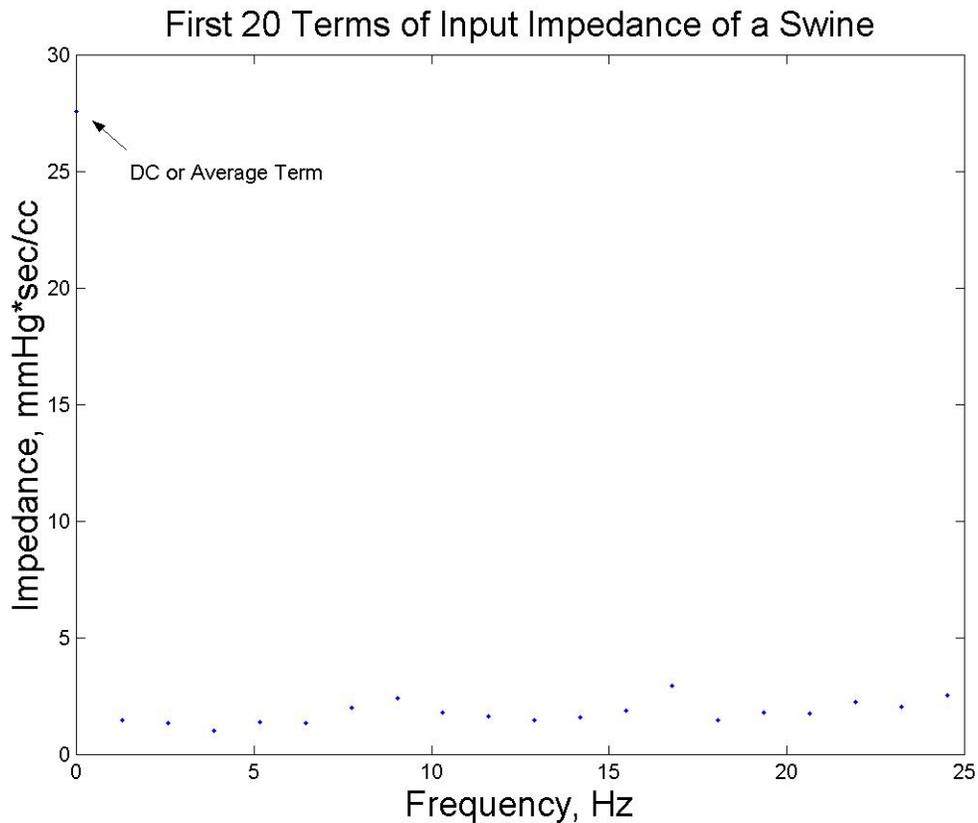
Time	AoP	AoP*cos(2*pi*f*t)	AoP*sin(2*pi*f*t)	AoP*cos(4*pi*f*t)	AoP*sin(4*pi*f*t)	→
0	65.0	65.0	0	65.0	0	...
0.005	65.1	65.1	2.7	64.9	5.3	...
0.010	65.7	65.5	5.4	64.8	10.7	...
0.015	67.0	66.6	8.2	65.1	16.3	...
0.020	68.8	67.9	11.2	65.2	22.1	...
0.025	70.6	69.1	14.3	64.7	28.0	...
...	...	...	...	...	...	...

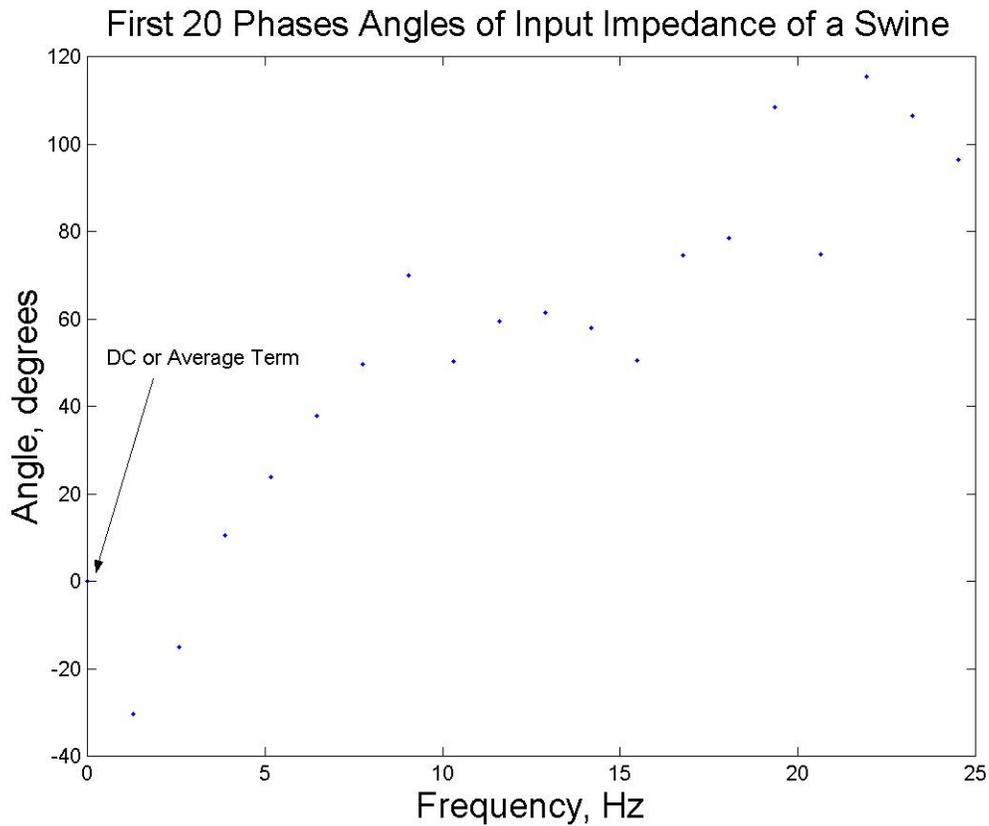


$$\begin{aligned}
 AoP(t) &= \overline{AoP} + A_1 \cos(2 * \pi * f * t) + B_1 \sin(2 * \pi * f * t) + A_2 \cos(4 * \pi * f * t) + B_2 \sin(4 * \pi * f * t) + \\
 &A_3 \cos(6 * \pi * f * t) + B_3 \sin(6 * \pi * f * t) + A_3 \cos(8 * \pi * f * t) + B_4 \sin(8 * \pi * f * t) + \\
 &A_5 \cos(10 * \pi * f * t) + B_5 \sin(10 * \pi * f * t) + A_6 \cos(12 * \pi * f * t) + B_6 \sin(12 * \pi * f * t) + \dots
 \end{aligned}$$

In summary, a technique has been given for decomposing a signal, such as aortic pressure, into an average and sum of sines and cosines. This technique is termed a Discrete Fourier Transform (DFT). Matlab can quickly perform this procedure by the command `fft`. The magnitudes are given by `abs(fft(AoP))` and the phases are obtained by `angle(fft(AoP))`. The magnitudes must be adjusted by dividing through by the number of samples (mean) and by one-half the number of samples (frequency terms).

OK, now once we have this method, what do we do with it? If we take the DFT Of AoP and AoF, we have the two inputs to the black box. If we take the first term of the DFT for AoP and divide by the first term of the DFT for the AoF, we have the clinical definition of TPR (average pressure divided by average flow). What do we get when we divide the respective second terms of the DFT? We get an impedance, which has units of resistance. Figure 8-7 shows typical input impedance magnitudes and phases of a swine. In most mammals, there is a large DC (average) term and then the magnitude drops off significantly at the first few harmonics. Clearly the angle drops negative and then it begins to rise to become positive.





**Figure 8-7 Magnitudes and Phases of first 20 Terms of Input Impedance of a Swine**

What information can we glean from the input impedance? First, we must understand basic electrical circuit concepts. We can write time-domain equations for resistors, capacitors, and inductors.

$$P_R = Q * R$$

$$Q_C = C \frac{dP_C}{dt}$$

$$P_L = L \frac{dQ_L}{dt}$$

For a system at steady state, we can write frequency domain equations relating pressure and flow for resistors, capacitors, and inductors using the Laplace transform.

$$P_R(s) = RQ(s)$$

$$P_C(s) = -j \frac{1}{\omega C} Q(s)$$

$$P_L(s) = j\omega L Q(s)$$

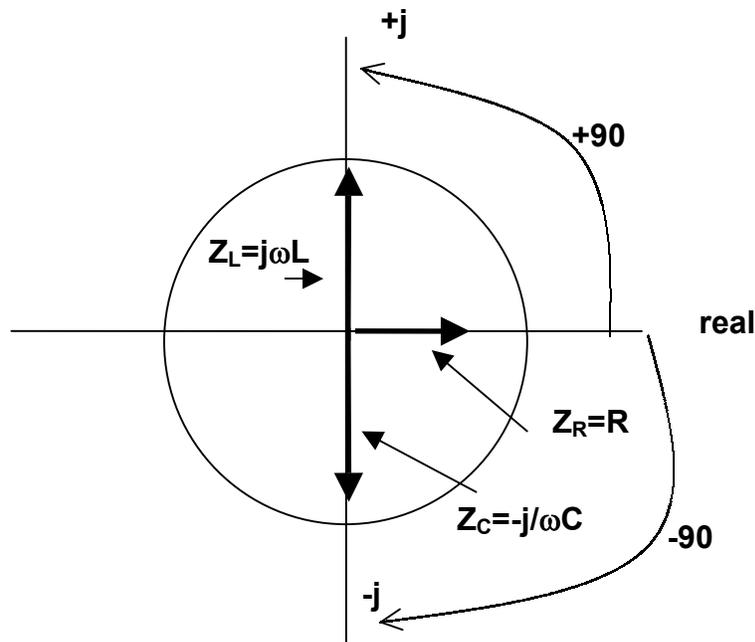
Thus, we can define the impedance of circuit elements as:

$$Z_R = R$$

$$Z_C = -j \frac{1}{\omega C}$$

$$Z_L = j\omega L$$

Thus, we can see that the impedance of an ideal resistance is not affected by frequency and there is no imaginary component to a resistor (zero phase angle). The impedance of a capacitor decreases with frequency and has a  $-90$  degree phase. The impedance of an inductor increases with frequency and has a phase of  $+90$  degrees. These components are shown on the unit circle in Figure 8-8.



**Figure 8-8 Phasor Representation of Impedances**

When we investigate the input impedance and see a decreasing magnitude and a negative phase at the fundamental frequency, we can surmise that capacitive and resistive effects are at work Figure 8-9.

As we continue to increase the frequency the phase angle begins to increase and it becomes positive, indicating the presence of an inductor. In hemodynamics inductance represents fluid inertia (inertia). Figure 8-10 depicts the phasor notation for the impedance at about 7 Hz (Figure 8-7)

As one can see there is much information about the circulatory system to be gleaned from the aortic input impedance. We can construct arterial circuit models that describe the frequency behavior seen in the impedance plot.

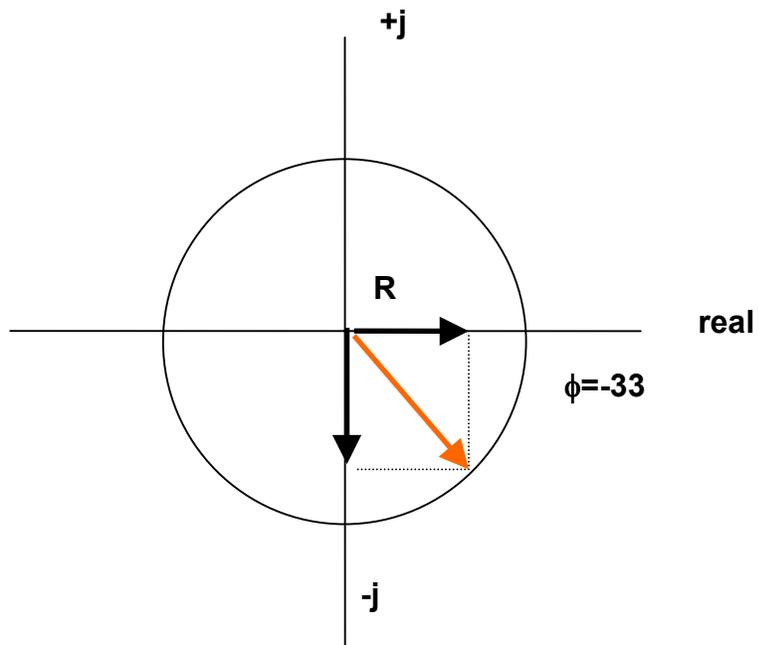


Figure 8-9 Phasor Representation of Impedance at Fundamental Frequency

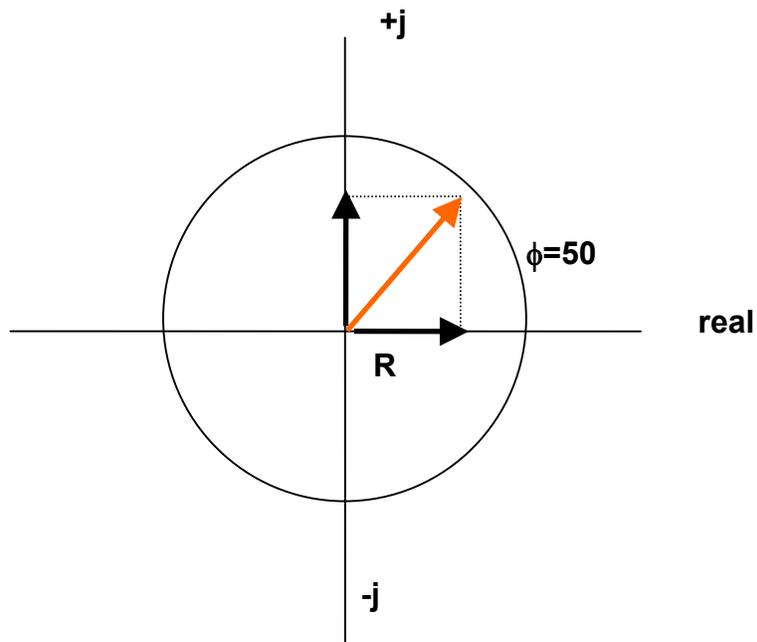
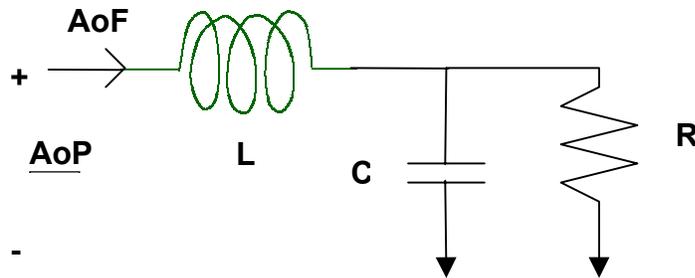


Figure 8-10 Phasor Representation of Impedance at Higher Frequency

At a minimum we can concoct a model with an inductor, capacitor and resistor that will begin to describe arterial system behavior at the lower frequencies, Figure 8-11.



**Figure 8-11 Simple Model of Arterial System**

For this model, at zero frequency (the first term of the input impedance) the impedance will be  $R$ . At lower frequencies the inductor will not play a role as ( $Z_L \approx 0$ ), but the capacitance will dominate (thus the negative phase). As the frequency increases, the inductor will become dominant and the phase will become positive.

How do we estimate the best values of  $R$ ,  $L$  and  $C$ ? We now return to `fminsearch` (or `fmins` for versions prior to 5.3) and calculate the best estimates. Here we take experimental `AoP` and `AoF`, obtain the DFT, and then try a circuit model to and iterate on the circuit elements to minimize the error between the experimental input impedance and that of the circuit model. The rationale for doing this is we can get estimates of arterial properties of resistance, compliance and inductance. Further, in the next chapter we'll use the cardiac and arterial models to investigate CV interaction.

**Parameter Estimation Technique - Again using our data set, we can estimate the R-L-C parameters in Matlab with the 'fminsearch' function.**

```
Three_el.m (not part of mfile)
global Zexp nharm w Zmod
AoF=AoF1*1000/60; % convert to cc/sec
fs=200; % sampling frequency in Hz
Zexp=fft(AoP1)./fft(AoF); % experimental input impedance
nharm=3; % the number of harmonics of interest
w=fs/length(AoP1) % the fundamental frequency, Hz
XX=fminsearch('Z_freqa',[1 1])% call fminsearch & initial cond [C
L]
Z_freqa.m
function error=Z_freqa(pp);
%this function minimizes error between experimental Z and model Z
global nharm w Zexp Zmod
%calculate Z model for specified harmonics for three element
inductance model
```

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```
harm=[1:nharm];
for ii=1:nharm-1
Z1(ii)=-j/(ii*w*pp(1)); %impedance of a capacitor
Z2(ii)=Zexp(1); %impedance of a resistor
Z3(ii)=j*ii*w*pp(2); %impedance of an inductor
ZA(ii)=Z1(ii)*Z2(ii)./(Z1(ii)+Z2(ii)); %impedance of cap and
%resistor in parallel
Zmod(ii)=ZA(ii)+Z3(ii); %ZA and inductor in series
real_e(ii)=(ii.^2)*(real(Zexp(ii+1))-real(Zmod(ii))); %wieghted
%error in the real component
imag_e(ii)=(ii.^2)*(imag(Zexp(ii+1))-imag(Zmod(ii))); %weighted
%error in the imaginary component
end
error=sum(real_e.^2)+sum(imag_e.^2);
AA=abs(Zmod);
BB=abs(Zexp);
plot(harm,[Zexp(1)AA(1:nharm-1)], '* ',harm,BB(1:nharm), '. ')
pause(.01)
```