

Rotation Matrices

A problem we'll be looking at in two days is trying to determine where the tip of a robot is. To do this, we use reference frames: an $\{X, Y, Z\}$ axis tied to each link of the robot. As the robot moves, often times the angle of each frame changes. Likewise, in order to go from one link to the next, a rotation and translation is required.

This lecture looks at rotation matrices.

In the $\{x, y, z\}$ plane, you can rotate about any or all axis. For simplicity, we will look at three separate rotations:

- Rotating about the X axis
- Rotating about the Y axis, and
- Rotating about the Z axis.

In addition, we will be referring to a point (P) relative to a specific reference frame. P_1 , for example, refers to the $\{x, y, z\}$ coordinate of point P relative to reference frame 1.

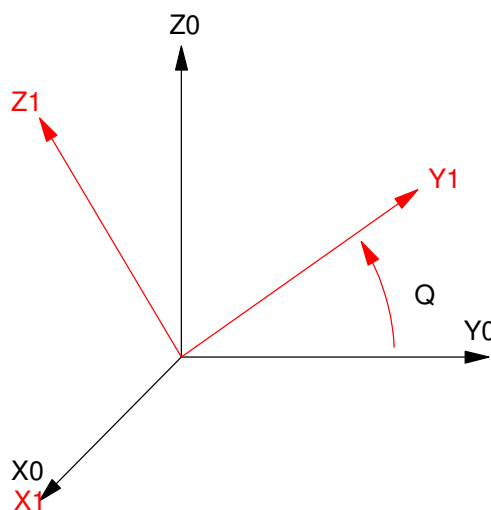
Rotation About the X Axis

Suppose you have a point P defined relative to reference frame 0 (P_0). Where is this point relative to reference frame 1 if reference frame 1 is equal to reference frame 0, rotated by θ about the x-axis?

Solution:

$$P_1 = T_{10} \cdot P_0$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

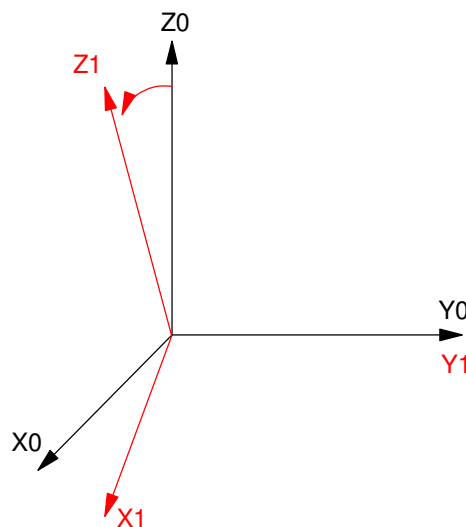


Rotate About the Y-Axis:

Repeat but rotate about the Y0 axis

$$P_1 = T_{10} \cdot P_0$$

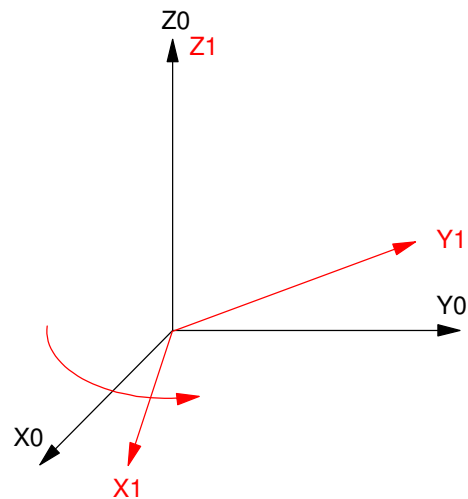
$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$



Rotate about the z-axis

$$P_1 = T_{10} \cdot P_0$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$



Example: A point relative to the zero reference frame is (1,2,3).

$$P_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

Where is it at if you

- Rotate 90 degrees about the X axis
- Rotate 90 degrees about the X axis
- Rotate 90 degrees about the X axis

Solution: For convenience, define c and s to be the sine and cosine of 90 degrees:

$$c = \cos(90 \cdot \pi / 180);$$

$$s = \sin(90 \cdot \pi / 180);$$

From the above equations

$$R_x = [1, 0, 0, 0; 0, c, s, 0; 0, -s, c, 0; 0, 0, 0, 1];$$

$$R_y = [c, 0, -s, 0; 0, 1, 0, 0; s, 0, c, 0; 0, 0, 0, 1];$$

$$R_z = [c, s, 0, 0; -s, c, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1];$$

$$P_0 = [1; 2; 3; 1]$$

$$R_x * P_0$$

$$R_y * P_0$$

$$R_z * P_0$$

No Rotation	X Axis	Y Axis	Z Axis
1	1	-3	2
2	3	2	-1
3	-2	1	3

Example 2: Where is the point if you

- Rotate 20 degrees about the Y axis, then
- Rotate 30 degrees about the Z axis?

Define three transformations. The point relative to reference frame 0 is

$$P_1 = T_{10} \cdot P_0 = \begin{bmatrix} \cos 20^\circ & 0 & \sin 20^\circ & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 20^\circ & 0 & \cos 20^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

$$P_2 = T_{21} \cdot P_1 = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P_1$$

or in MATLAB

```
c = cos(20*pi/180);
s = sin(20*pi/180);

Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];

T10 = Ry;

c = cos(30*pi/180);
s = sin(30*pi/180);

Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];
Ry = [c,0,-s,0;0,1,0,0;s,0,c,0;0,0,0,1];
Rz = [c,s,0,0;-s,c,0,0;0,0,1,0;0,0,0,1];

T21 = Rz;

P0 = [1;2;3;1]
P1 = T10 * P0
P2 = T21 * P1
```

The point relative to the zero reference frame is:

$P_0 =$

1
2
3
1

The point relative to the first reference frame (rotated 20 degrees about the y axis)

$P_1 =$

-0.0864
2.0000
3.1611
1.0000

The point relative to the second reference frame (finally rotated 30 degrees about the z axis)

$P_2 =$

0.9252
1.7752
3.1611
1.0000

In the rotated coordinate system, the point is located at $P_2 = \begin{pmatrix} 0.9252 \\ 1.7752 \\ 3.1611 \end{pmatrix}$.

Displaying a 3D Object

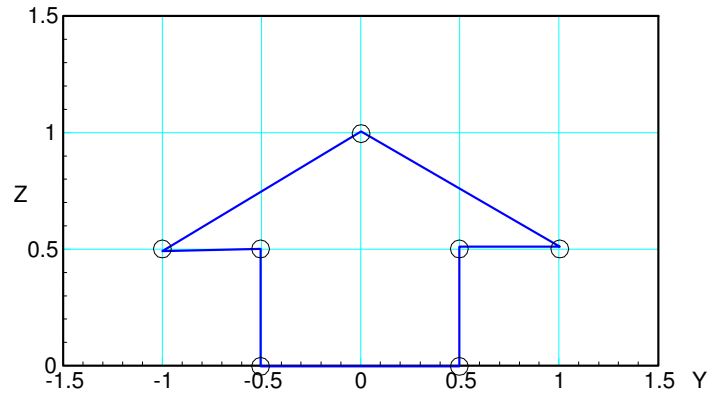
To illustrate the use of rotation matrices, let's draw an arrow and then

- Rotate the camera about the X, Y, and Z axis, then
- Rotate the arrow about the X, Y, and Z axis

Define the arrow by eight points:

$$Arrow = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0.5 & 1 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0 \end{bmatrix}$$

Plotted on the YZ plans looks like this:



Since the screen is 2-dimensional, project the arrow on the YZ plane. An m-file to display a set of points (passed in DATA) along with a transformation matrix is as follows:

```
function Display3D(DATA, T)

% scaling factor
s = T(4,4);

% draw the X, Y, Z axis
X = [1,0,0,1]';
Y = [0,1,0,1]';
Z = [0,0,1,1]';
O = [0,0,0,1]';

DATA = T*DATA;

T0 = T;
T0(1,4) = 0;
T0(2,4) = 0;
T0(3,4) = 0;

% transform
X0 = T0*X;
Y0 = T0*Y;
Z0 = T0*Z;
Origin = T0*O;

% display is the y-z plane
hold off;
plot([-2,2],[-2,2],'wx');
hold on;

% Project onto the YZ axis

Tx = s*[0,1,0,0];
Ty = s*[0,0,1,0];

plot(Tx*[Origin, X0], Ty*[Origin,X0], 'g');
plot(Tx*[Origin, Y0], Ty*[Origin,Y0], 'r');
plot(Tx*[Origin, Z0], Ty*[Origin,Z0], 'm');

% display the data

plot(Tx*DATA,Ty*DATA, 'b')

end
```

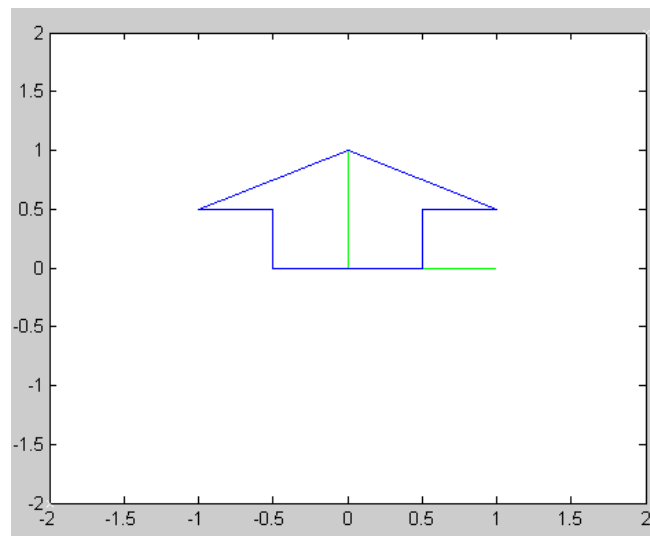
The previous arrow looks like the following:

```
X = [0,0,0,0,0,0,0,0]';  
Y = [-0.5,0.5,0.5,1,0,-1,-0.5,-0.5]';  
Z = [0,0,0.5,0.5,1,0.5,0.5,0]';
```

```
ARROW = [X,Y,Z,0*X+1]'
```

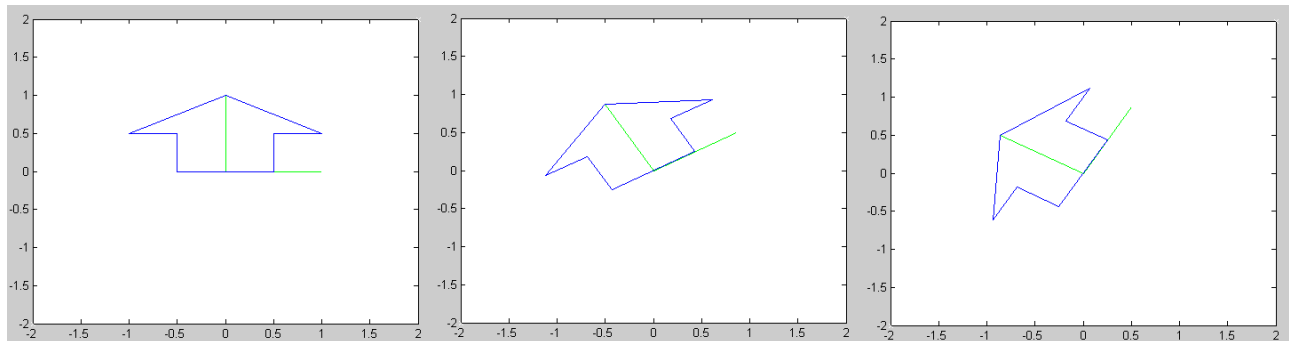
```
      0      0      0      0      0      0      0      0  
-0.5000  0.5000  0.5000  1.0000  0      -1.0000  -0.5000  -0.5000  
      0      0      0.5000  0.5000  1.0000  0.5000  0.5000  0  
      1.0000  1.0000  1.0000  1.0000  1.0000  1.0000  1.0000  1.0000
```

```
T = eye(4,4);  
Display3D(ARROW, T);
```



Rotate the camera about the X axis

```
c = cos(5*pi/180);  
s = sin(5*pi/180);  
Rx = [1,0,0,0;0,c,s,0;0,-s,c,0;0,0,0,1];  
  
T = eye(4,4);  
for i=1:1000  
    T = Rx*T;  
    Display3D(ARROW, T);  
    pause(0.01);  
end
```



Fixed Arrow, Camera Rotated 0, -30, and -60 degrees about the X axis

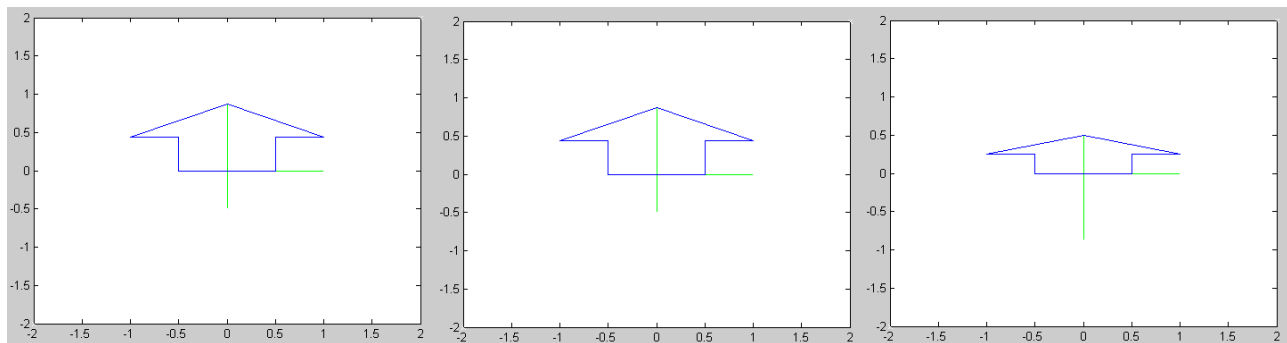
Note that the camera spins counter-clockwise (positive direction). This makes it look like the arrow is spinning counter-clockwise

Rotate the Camera About the Y Axis

```
Ry = [c, 0, -s, 0; 0, 1, 0, 0; s, 0, c, 0; 0, 0, 0, 1];
```

```
    0.9962      0      0.0872      0
      0      1.0000      0      0
   -0.0872      0      0.9962      0
      0      0      0      1.0000
```

```
T = eye(4,4);
for i=1:1000
    T = Ry*T;
    Display3D(ARROW, T);
    pause(0.01);
end
```



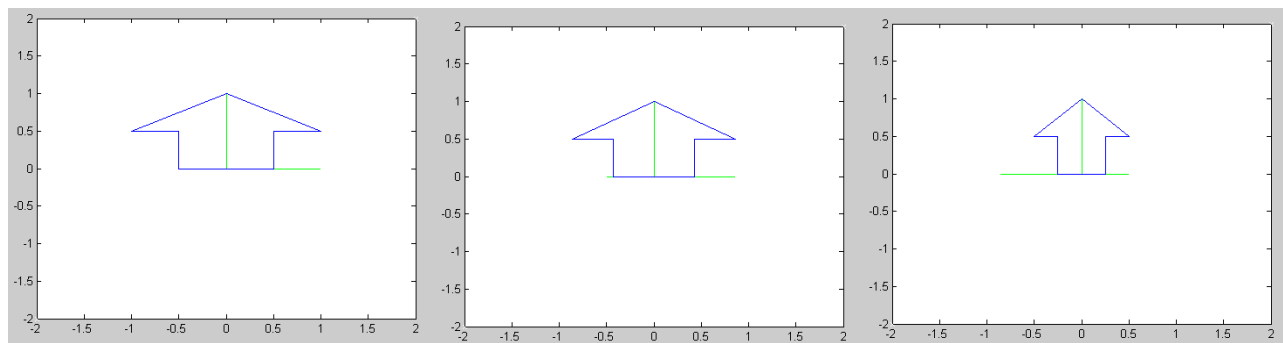
Fixed Arrow, Camera Rotated 0, 30, and 60 degrees about the Y axis

Rotate the Camera About the Z Axis

```
Rz = [c, -s, 0, 0; s, c, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]
```

```
0.9962    -0.0872         0         0  
0.0872     0.9962         0         0  
0           0         1.0000         0  
0           0           0         1.0000
```

```
T = eye(4,4);  
for i=1:1000  
    T = Rz*T;  
    Display3D(ARROW, T);  
    pause(0.01);  
end
```



Fixed Arrow, Camera Rotated 0, 30, and 60 degrees about the Z axis

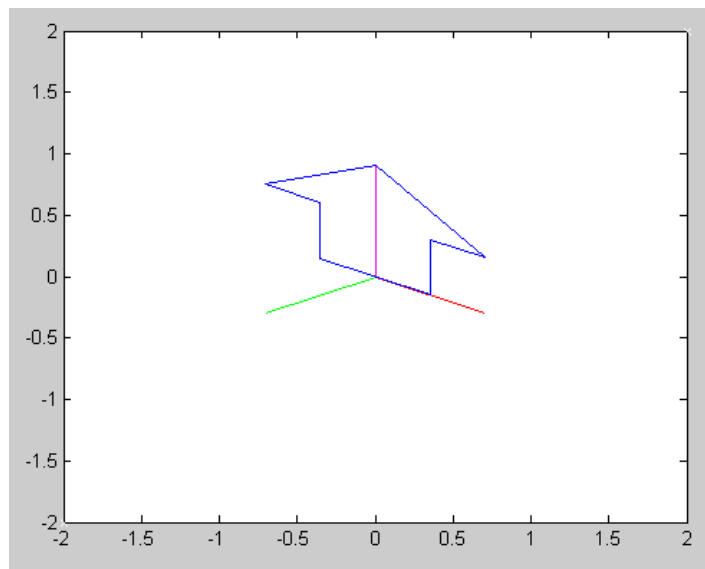
3D Perspective:

Move the camera 45 degrees about the Z axis then 25 degrees about the Y axis

```
c = cos(-25*pi/180);  
s = sin(-25*pi/180);  
Ry = [c, 0, -s, 0; 0, 1, 0, 0; s, 0, c, 0; 0, 0, 0, 1];  
c = cos(-45*pi/180);  
s = sin(-45*pi/180);  
Rz = [c, -s, 0, 0; s, c, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]  
Tdisp = Ry*Rz
```

```
    0.6409    0.6409    0.4226    0  
   -0.7071    0.7071    0        0  
   -0.2988   -0.2988    0.9063    0  
    0         0         0        1.0000
```

```
Display3D(ARROW, Tdisp);
```



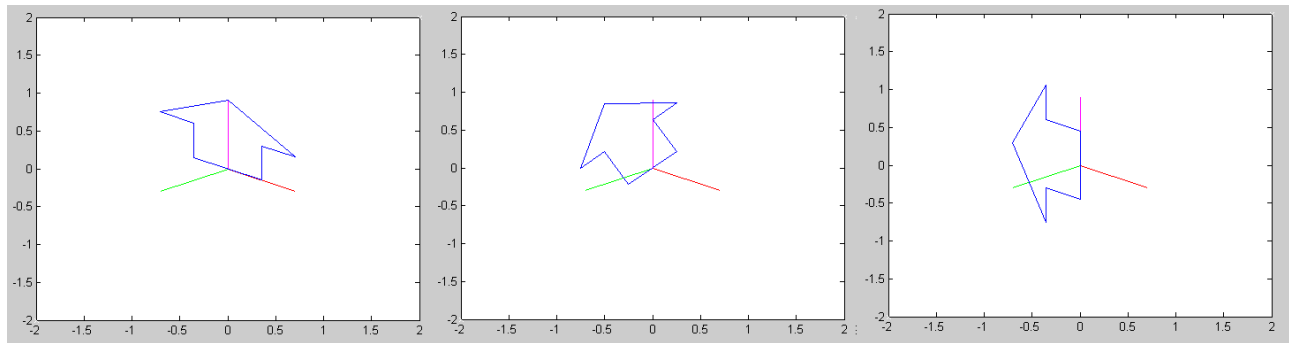
Viewing the arrow on the YX plane from an angle of Rotate Z 45 deg then Rotate Y 45 deg

Rotate the Arrow About the X Axis

You can also rotate the object while keeping the axis fixed

```
c = cos(5*pi/180);
s = sin(5*pi/180);
Rx = [1, 0, 0, 0; 0, c, s, 0; 0, -s, c, 0; 0, 0, 0, 1];
T01 = inv(Rx);

T = eye(4, 4);
for i=1:1000
    T = T01*T;
    Display3D(T*ARROW, Tdisp);
    pause(0.01);
end
```



Rotate the Object by 0, 45, and 90 degrees about the X axis

Note: The arrow positions are relative to reference frame 1:

$$P_1$$

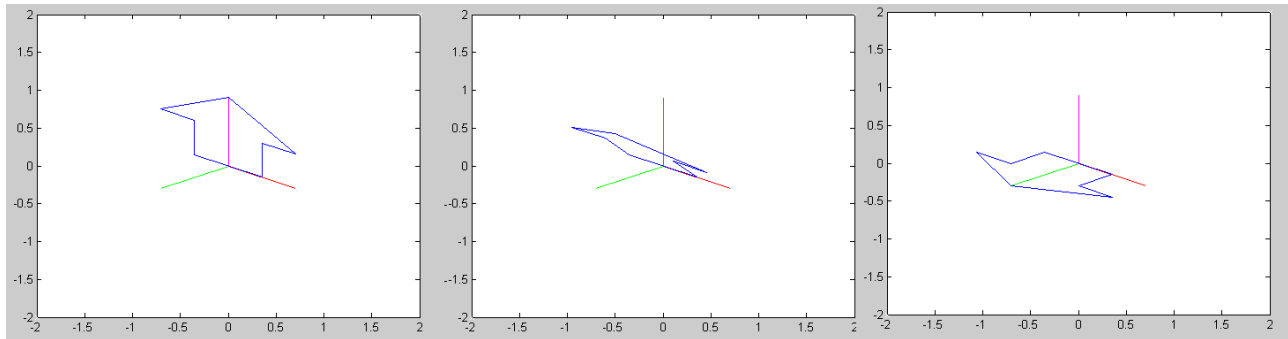
Relative to the zero reference frame, the points are:

$$P_0 = T_{01}P_1$$

$$P_0 = (T_{10})^{-1}P_1$$

Rotate the Object About the Y Axis

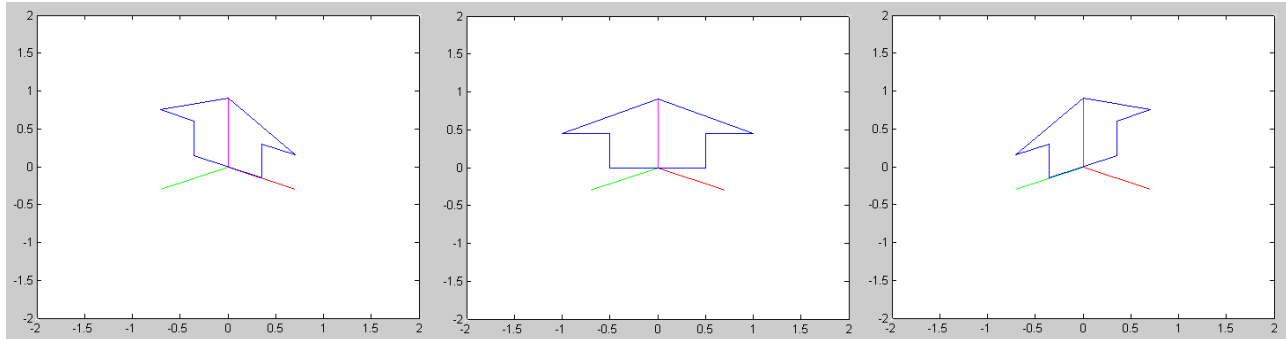
```
c = cos(5*pi/180);  
s = sin(5*pi/180);  
Ry = [c, 0, -s, 0; 0, 1, 0, 0; s, 0, c, 0; 0, 0, 0, 1];  
T01 = inv(Ry);  
  
T = eye(4,4);  
for i=1:1000  
    T = T01*T;  
    Display3D(T*ARROW, Tdisp);  
    pause(0.01);  
end
```



Rotate the Object 0, 45, and 90 degrees about the Y axis

Rotate the Object About the Z Axis

```
c = cos(5*pi/180);  
s = sin(5*pi/180);  
Rz = [c, s, 0, 0; -s, c, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1];  
T01 = inv(Rz);  
  
T = eye(4, 4);  
for i=1:1000  
    T = T01*T;  
    Display3D(T*ARROW, Tdisp);  
    pause(0.01);  
end
```



Rotate the Object 0, 45, and 90 Degrees About the Z Axis