LaGrangian Formulation of System Dynamics

Find the dynamics of a nonlinear system:

Circuit analysis tools work for simple lumped systems. For more complex systems, especially nonlinear ones, this approach fails. The Lagrangian formulation for system dynamics is a way to deal with any system.

Definitions:

- **KE**  Kinetic Energy in the system
- **PE**  Potential Energy
- \( \frac{\partial}{\partial t} \)  The partial derivative with respect to 't'. All other variables are treated as constants.
- \( \frac{d}{dt} \)  The full derivative with respect to t.
\[
\frac{d}{dt} = \frac{\partial}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial}{\partial z} \frac{\partial z}{\partial t} + \ldots
\]
- **L**  Lagrangian = KE - PE

Procedure:

1) Define the kinetic and potential energy in the system.
2) Form the Lagrangian:
\[
L = KE - PE
\]
3) The input is then
\[
F_i = \frac{d}{dt} \left( \frac{\partial L}{\partial x_i} \right) - \frac{\partial L}{\partial x_i}
\]
where \( F_i \) is the input to state \( x_i \). Note that
- If \( x_i \) is a position, \( F_i \) is a force.
- If \( x_i \) is an angle, \( F_i \) is a torque
Example:

Example: Determine the dynamics of a rocket

Step 1: Determine the potential and kinetic energy of the rocket

Potential Energy

\[ PE = mgx \]

Kinetic Energy:

\[ KE = \frac{1}{2}m\dot{x}^2 \]

Step 2: Set up the LaGrangian

\[ L = KE - PE \]

\[ L = \frac{1}{2}m\dot{x}^2 - mgx \]

Step 3: Take the partials

\[ F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) \]

\[ F = \frac{d}{dt}(m\dot{x}) - (-mg) \]

Take the full derivative with respect to t

\[ F = m\ddot{x} + \dot{m}\dot{x} + mg \]

Note that if the rocket is losing mass you get the term \( \dot{m}\dot{x} \). If you leave this term out, the rocket misses the target.
Example 2: Ball in a parabolic bowl

Determine the dynamics of a ball rolling in a bowl characterized by

$$y = \frac{1}{2}x^2$$

Step 1: Define the kinetic and potential energy

Potential Energy:

$$PE = mgy = \frac{1}{2}mgx^2$$

Kinetic Energy: This has two terms, one for translation and one for rotation.

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}J\dot{\theta}^2$$

The velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

The rotational velocity is

$$position = r\theta$$

$$v = r\dot{\theta}$$

Note that

$$y = \frac{1}{2}x^2$$

$$\dot{y} = x\ddot{x}$$

gives
\[ KE = \frac{1}{2} mv^2 + \frac{1}{2} J(\dot{\phi})^2 \]
\[ KE = \frac{1}{2} \left( m + \frac{J}{r^2} \right) v^2 \]
\[ KE = \frac{1}{2} \left( m + \frac{J}{r^2} \right) (\dot{x}^2 + \dot{y}^2) \]
\[ KE = \frac{1}{2} \left( m + \frac{J}{r^2} \right) (\dot{x}^2 + (x\dot{x})^2) \]

The inertia depends upon what type of ball you are using:

- \( J = 0 \) point mass with all the mass in the center
- \( J = \frac{2}{5} mr^2 \) solid sphere
- \( J = \frac{2}{3} mr^2 \) hollow sphere
- \( J = mr^2 \) hollow cylinder

Assume the ball is a solid sphere

\[ KE = \frac{1}{2} \left( m + \frac{2}{5} mr^2 \right) (\dot{x}^2 + (x\dot{x})^2) \]
\[ KE = 0.7m(1^2 + x^2)\dot{x}^2 \]

Step 2: Form the LaGrangian

\[ L = KE - PE \]
\[ L = 0.7m(1^2 + x^2)\dot{x}^2 - \frac{1}{2}mgx^2 \]

Step 3: Take the partials. The partial with respect to \( x \) is:

\[ \frac{\partial L}{\partial x} = 0.7m(2x)\dot{x}^2 - mgx \]
\[ \frac{\partial L}{\partial x} = 1.4m\dot{x}^2 - mgx \]

The partial with respect to \( dx/dt \) is:

\[ \frac{\partial L}{\partial x} = 1.4m(1^2 + x^2)\dot{x} \]
The full derivative of the partial with respect to $\frac{dx}{dt}$ is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{d}{dt} (1.4m(1^2 + x^2) \dot{x})$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial x} \right) = 1.4m(2\ddot{x}) \dot{x} + 1.4m(1^2 + x^2) \ddot{x}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 2.8m \dot{x}^2 + 1.4m(1^2 + x^2) \ddot{x}$$

So, the dynamics are:

$$F = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \left( \frac{\partial L}{\partial x} \right)$$

$$F = (2.8m \dot{x}^2 + 1.4m(1^2 + x^2) \dot{x}) - (1.4m \dot{x}^2 - mg)$$

$$F = 1.4m \dot{x}^2 + 1.4m(1^2 + x^2) \ddot{x} + mg$$

In free fall, $F = 0$. Solving for the highest derivative:

$$\ddot{x} = -\left( \frac{1.4 \dot{x}^2 + g}{1.4(1^2 + x^2)} \right)$$
Matlab Code (Ball.m)

```matlab
% Dynamics of a ball rolling in a bowl where
%    y = 1/2 x^2
%
x = 1.5;
dx = 0;
dt = 0.01;
t = 0;
while(t < 100)

% compute the acceleration
ddx = -( 1.4*dx*dx + 9.8) * x / ( 1.4*(1 + x*x) );

% integrate
dx = dx + ddx*dt;
x = x + dx*dt;

% display the ball
y = 0.5*x*x;

x1 = [-2:0.01:2]';
y1 = 0.5* (x1 .^ 2);

% draw the ball
i = [0:0.01:1]' * 2 * pi;
xb = 0.05*cos(i) + x;
yb = 0.05*sin(i) + 0.5*x^2 + 0.05 + 0.02*abs(x);

% line through the ball
q = [0, pi] - x/0.05;
xb1 = 0.05*cos(q) + x;
yb1 = 0.05*sin(q) + 0.5*x^2 + 0.05 + 0.02*abs(x);

plot(x1,y1,'b', xb, yb, 'r', xb1, yb1, 'r');
pause(0.01);
end
```