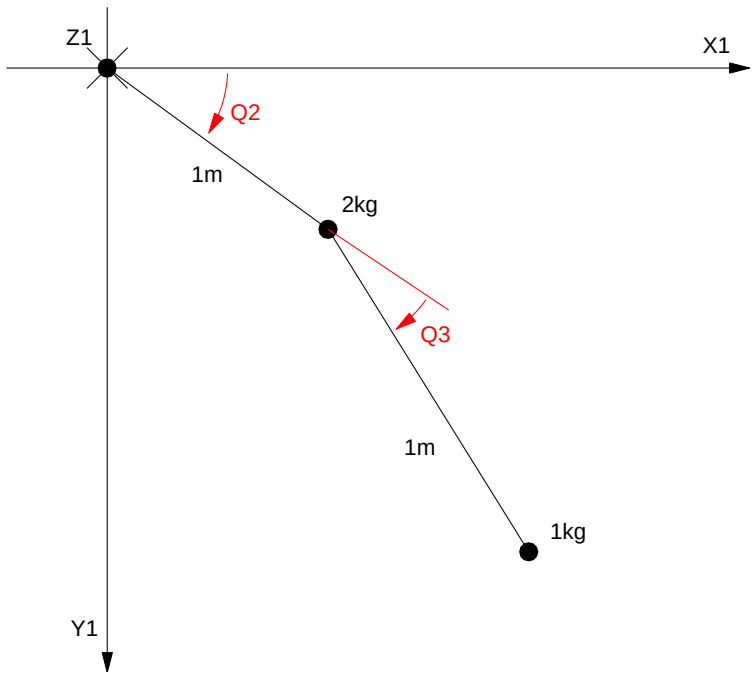
Dynamics of a 2-Link Arm

Consider a planar 2-link robot as shown below. This is similar to the RRR robot with $\theta_1 = 0$ resulting in the robot being able to move in the XY plane relative to reference frame 1.



Problem: Determine the dynamics of a 2-link arm with

- Each arm having a length of 1m, and
- Mass #1 = 2kg and Mass #2 = 1kg.

Step 1: Determine the (x,y) position and velocity of each mass

mass 1:

$$x_1 = -\cos\theta_1 \qquad \dot{x}_1 = -\sin(\theta_1)\dot{\theta}_1$$

$$y_1 = \sin(\theta_1) \qquad \dot{y}_1 = \cos(\theta_1)\dot{\theta}_1$$

mass 2:

$$x_2 = x_1 + \cos(\theta_1 + \theta_2) \qquad \dot{x}_2 = -\sin(\theta_1)\dot{\theta}_1 - \sin(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = y_1 + \sin(\theta_1 + \theta_2) \qquad \dot{y}_2 = \cos(\theta_1)\dot{\theta}_1 + \cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

Using shorthand notation

$$x_1 = c_1 \qquad \dot{x}_1 = -s_1\dot{\theta}_1$$

$$y_1 = s_1 \qquad \dot{y}_1 = c_1\dot{\theta}_1$$

$$x_2 = c_1 + c_{12} \qquad \dot{x}_2 = -s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$y_2 = s_1 + s_{12} \qquad \dot{y}_2 = c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

Step 2: Form the kinetic and potential energy:

Mass 1: $m_1 = 2$

$$KE = \tfrac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$KE = \left(\left(-s_1\dot{\theta}_1\right)^2 + \left(c_1\dot{\theta}_1\right)^2\right)$$

Note that $\sin^2 + \cos^2 = 1$

$$KE = \dot{\theta}_1^2$$

$$PE = m_1 g y_1$$

$$PE = -2gs_1$$

Mass 2: $m_2 = 1$

$$KE = \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$KE = \tfrac{1}{2}\left(\left(-s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2 + \left(c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2\right)$$

$$KE = \tfrac{1}{2}\left(s_1^2\dot{\theta}_1^2 + s_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + c_1^2\dot{\theta}_1^2 + c_{12}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right)$$

$$+s_1 s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_1 c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = \tfrac{1}{2}\left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) + c_1 c_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + s_1 s_{12}\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

From trigenomietry

$$\sin(a)\sin(b) + \cos(a)\cos(b) = \cos(a - b)$$

resulting in

$$KE = \tfrac{1}{2}\left(\dot{\theta}_1^2 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$KE = (1 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2$$

$$PE = m_2 g y_2$$

$$PE = -g(s_1 + s_{12})$$

Step 3: Form the LaGrangian:

$$L = KE - PE$$

$$L = \left(\left(\dot{\theta}_1^2\right) + \left((1 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2\right)\right) - ((-2gs_1) + (-g(s_1 + s_{12})))$$

$$\boxed{L = (2 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 + 3gs_1 + gs_{12}}$$

Step 4: Take the partials

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

Starting with θ_1:

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right)$$

$$T_1 = \frac{d}{dt}\left((4 + 2c_2)\dot{\theta}_1 + (1 + c_2)\dot{\theta}_2\right) - (3gc_1 + gc_{12})$$

$$\boxed{T_1 = \left((4 + 2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1 + c_2)\ddot{\theta}_2 - s_2\dot{\theta}_2^2\right) - (3gc_1 + gc_{12})}$$

Moving on to θ_2

$$L = (2 + c_2)\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + (1 + c_2)\dot{\theta}_1\dot{\theta}_2 + 3gs_1 + gs_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right)$$

$$T_2 = \frac{d}{dt}\left(\dot{\theta}_2 + (1 + c_2)\dot{\theta}_1\right) - \left(-s_2\dot{\theta}_1^2 - s_2\dot{\theta}_1\dot{\theta}_2 + gc_{12}\right)$$

$$T_2 = \left(\ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2\right) + \left(s_2\dot{\theta}_1^2 + s_2\dot{\theta}_1\dot{\theta}_2 - gc_{12}\right)$$

$$\boxed{T_2 = \ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 - gc_{12}}$$

Net result:

$$T_1 = \left((4 + 2c_2)\ddot{\theta}_1 - 2s_2\dot{\theta}_1\dot{\theta}_2 + (1 + c_2)\ddot{\theta}_2 - s_2\dot{\theta}_2^2\right) - (3gc_1 + gc_{12})$$

$$T_2 = \ddot{\theta}_2 + (1 + c_2)\ddot{\theta}_1 + s_2\dot{\theta}_1^2 - gc_{12}$$

To solve, rewrite this in matrix form:

$$\begin{bmatrix} (4 + 2c_2) & (1 + c_2) \\ (1 + c_2) & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + g\begin{bmatrix} 3c_1 + c_{12} \\ c_{12} \end{bmatrix}$$

Mass Matrix * Acceleration = Torque - Coriolis Forces + Gravity

Given the joint angles, velocities, gravity, and input torques, you can compute the joint accelerations as

$$\boxed{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} (4 + 2c_2) & (1 + c_2) \\ (1 + c_2) & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 2s_2\dot{\theta}_1\dot{\theta}_2 + s_2\dot{\theta}_2^2 \\ -s_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 3gs_1 + gs_{12} \\ gs_{12} \end{bmatrix}\right)}$$

MatLab Code:

```
function [ ddQ ] = TwoLinkDynamics( Q, dQ, T )
q1 = Q(1);
q2 = Q(2);
dq1 = dQ(1);
dq2 = dQ(2);
c1 = cos(q1);
s1 = sin(q1);
s2 = sin(q2);
c2 = cos(q2);
s12 = sin(q1+q2);
c12 = cos(q1+q2);
g = 9.8;
M = [ 4 + 2*c2 , 1+c2 ;  1+c2 , 1];
C = [ 2*s2*dq1*dq2 + s2*dq2*dq2 ;   -s2*dq1*dq1 ];
G = [3*g*c1 + g*c12 ;   +g*c12 ];
ddQ = inv(M) * ( T + C + G );
end
```

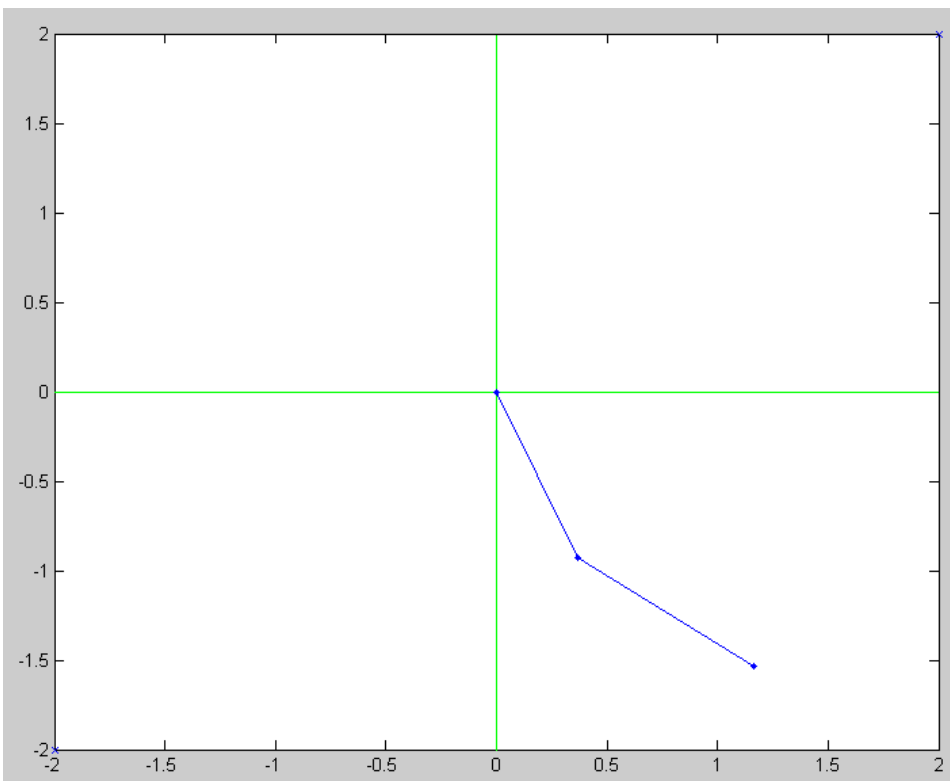


Image of 2-link arm in free-fall

Main Calling Routine:

```
Q = [0; 0];
dQ = [0; 0];
T = [0; 0];
t = 0;
dt = 0.01;

while(1)

    c1 = cos(Q(1));
    s1 = sin(Q(1));
    c2 = cos(Q(2));
    s2 = sin(Q(2));

% Freefall
    T = [0;0];

    ddQ = TwoLinkDynamics(Q, dQ, T);
    dQ = dQ + ddQ * dt;
    Q = Q + dQ*dt;
    t = t + dt;

 % Plot the Robot
    x0 = 0;
    y0 = 0;
    x1 = cos(Q(1));
    y1 = sin(Q(1));
    x2 = x1 + cos(Q(1) + Q(2));
    y2 = y1 + sin(Q(1) + Q(2));

    clf;
    plot([-0.5,2],[-0.5,2],'x');
    hold on;
    plot([-0.5,2],[0,0], 'r');
    plot([0,0],[-0.5,2], 'r');
    plot([x0, x1, x2], -[y0, y1, y2], 'b.-');
    pause(0.01);

    end
```